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More on 2-Bundles with 2-Connections

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Abstract: A collection of some material related to the theory of 2-bundles with 2-connections.

Contents

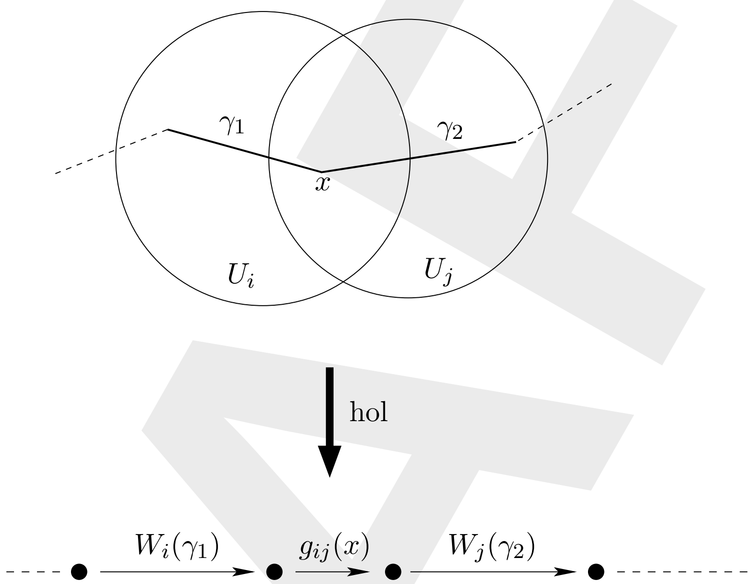
1. Introduction

This is a private set of sketchy notes containing material related to 2-bundles with 2-connections [1, 2].

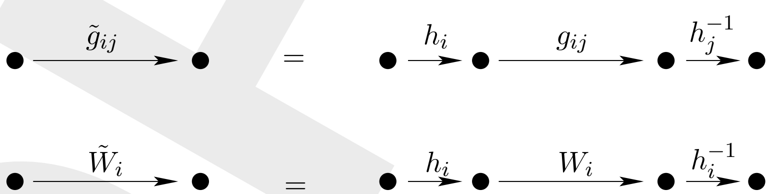
2. 2-Gauge Transformations and global 2-Holonomy

2.1 1-Gauge Transformations and global 1-Holonomy

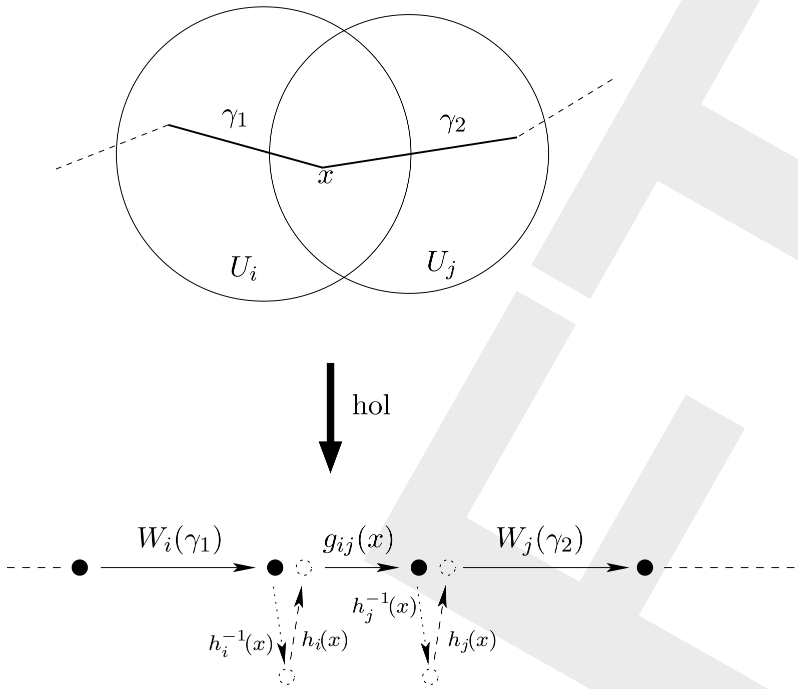
Global line holonomy in a fiber bundle is computed this way:



A gauge transformation in the locally trivialized bundle amounts to



Plugging this into the definition of the global line holonomy one finds its invariance under gauge transformations, as it should be:



These simple facts have a natural generalization when going from ordinary bundles to 2-bundles. This is discussed in the following subsections.

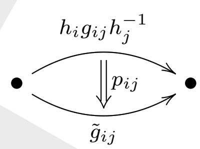
2.2 2-Gauge Transformations

The derivation of the nature of 2-gauge transformations will be postponed until §2.6 (p.14). For now we just state the result and then use it to define global 2-holonomy.

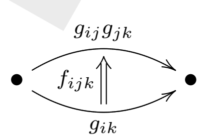
Throughout we will consider two fixed but arbitrary gauges (i.e. two choices of local 2-trivialization), denoted G and $\tilde{G}$. Objects in the gauge $\tilde{G}$ will be denoted by the corresponding symbol in the gauge G but decorated with a $\tilde{\cdot}$.

(Currently the entire discussion applies to 2-bundles with *trivial* base 2-spaces only. The generalization of all of the following to nontrivial base 2-spaces should be straightforward but is not yet considered here.)

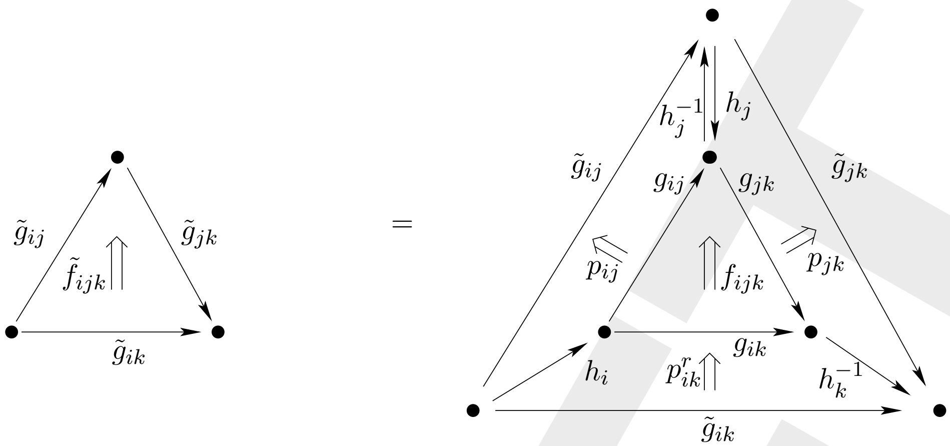
The transition 2-map g has the gauge transformation



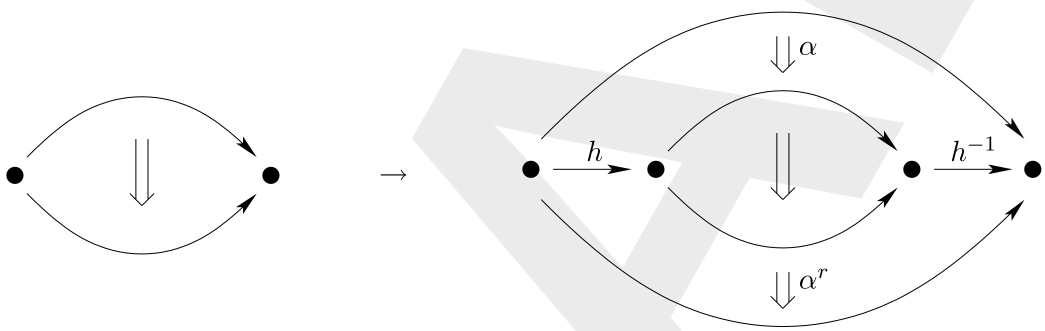
This implies for the f_{ijk}



the transformation

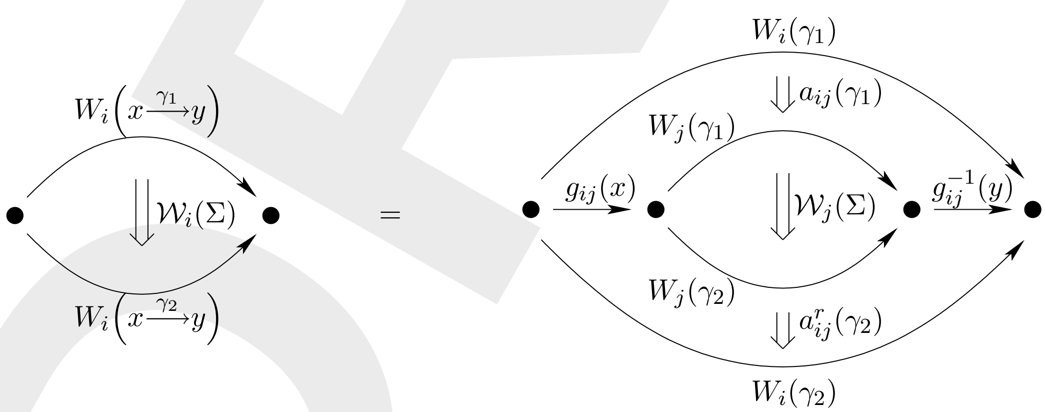


On the other hand, the 2-holonomies transform under *2-conjugations*, by which we mean operations on 2-group elements of the form

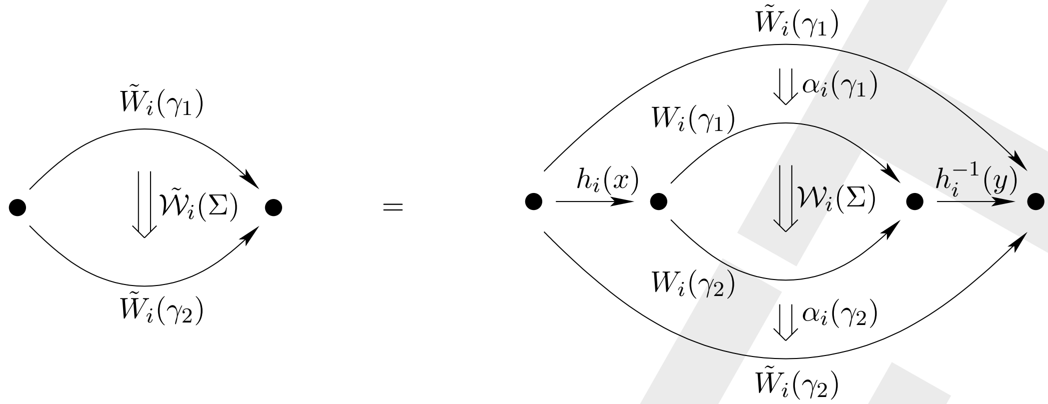


Indeed, the known [2] transition law for the 2-holonomy

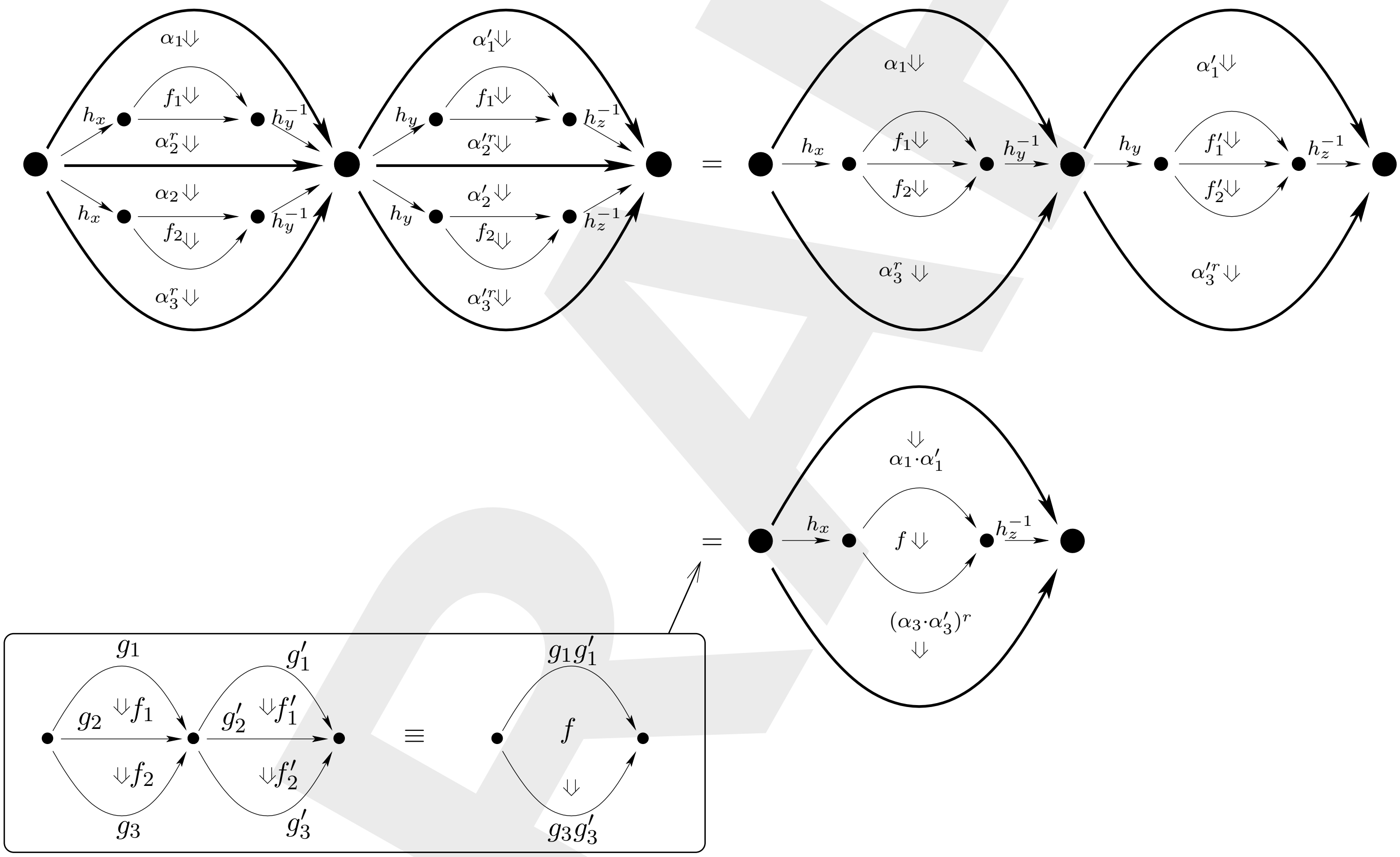
In fact there this was derived only for closed loops. The generalization follows immediately but still needs to be discussed.



is of this form. One finds that the gauge transformation of the 2-holonomy is given by



These 2-conjugations respect 2-dimensional composition in that

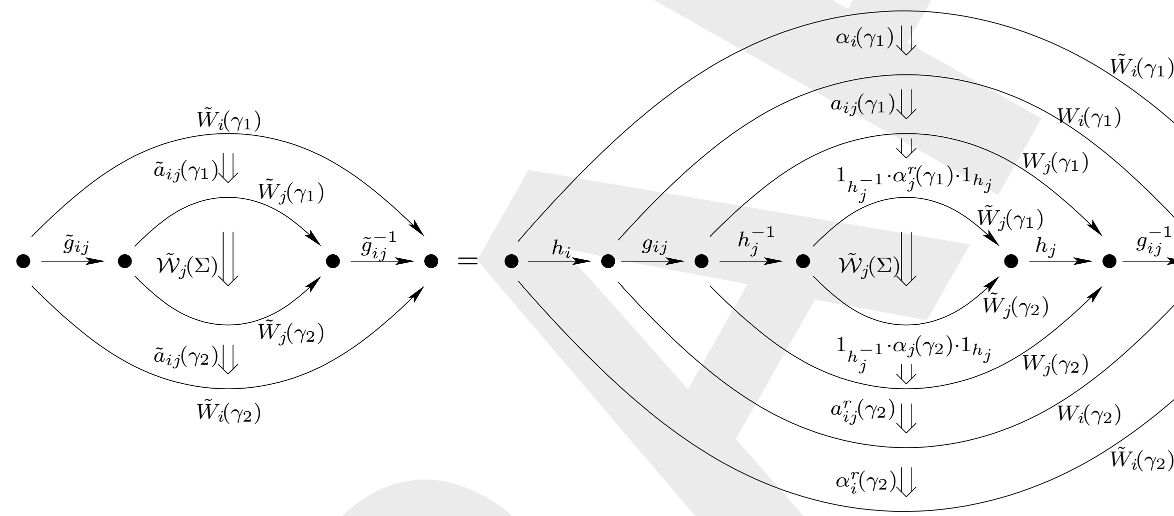


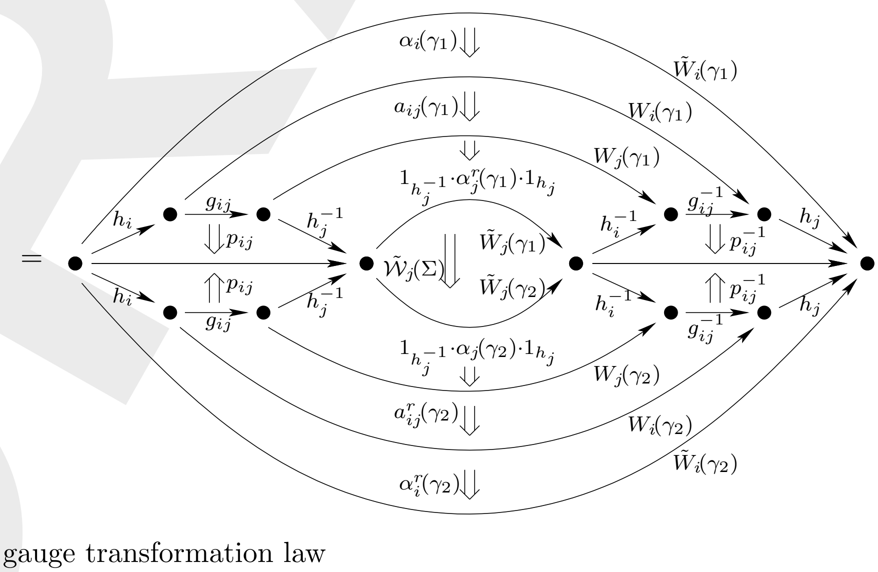
We display this rather simple fact in such a detail because it gives a good illustration of the way how in a composite diagram of gauge transformed quantities 2-conjugation operations mutually cancel and leave the diagrams in the original gauge behind. The proof of the gauge invariance of global 2-holonomy in §2.4 (p.10) proceeds in precisely this fashion.

Like the gauge transformation of the transition map g induces a gauge transformation for f, the above gauge transformation for the 2-holonomy $\mathcal{W}$ induces a gauge transformation for the a_{ij}

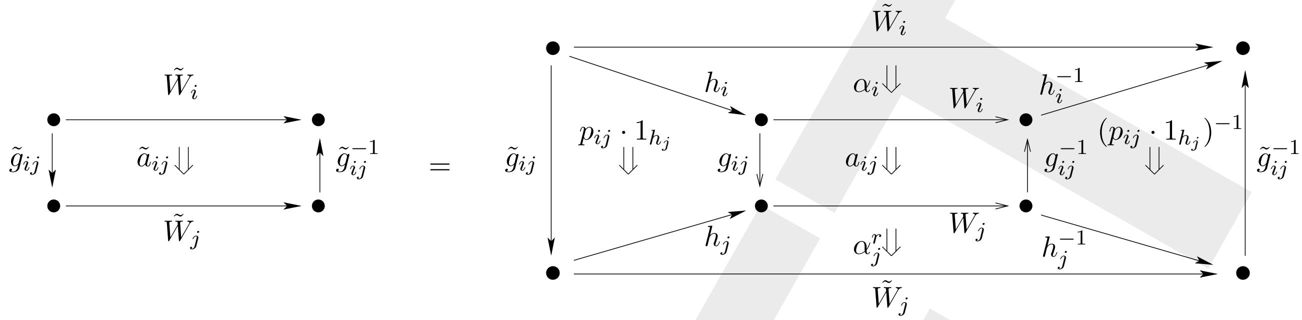
$$\begin{array}{ccc} \bullet & \xrightarrow{\;W_i\left(x\xrightarrow{\gamma}y\right)\;} & \bullet \\ {\scriptstyle g_{ij}(x)}\downarrow & \Downarrow\; a_{ij}\left(x\xrightarrow{\gamma}y\right) & \uparrow{\scriptstyle g_{ij}^{-1}(y)} \\ \bullet & \xrightarrow[\;W_j\left(x\xrightarrow{\gamma}y\right)\;]{} & \bullet \end{array} \tag{2.1}$$

This is obtained by considering a transition first in one gauge and then in the other:





From this one reads off the gauge transformation law



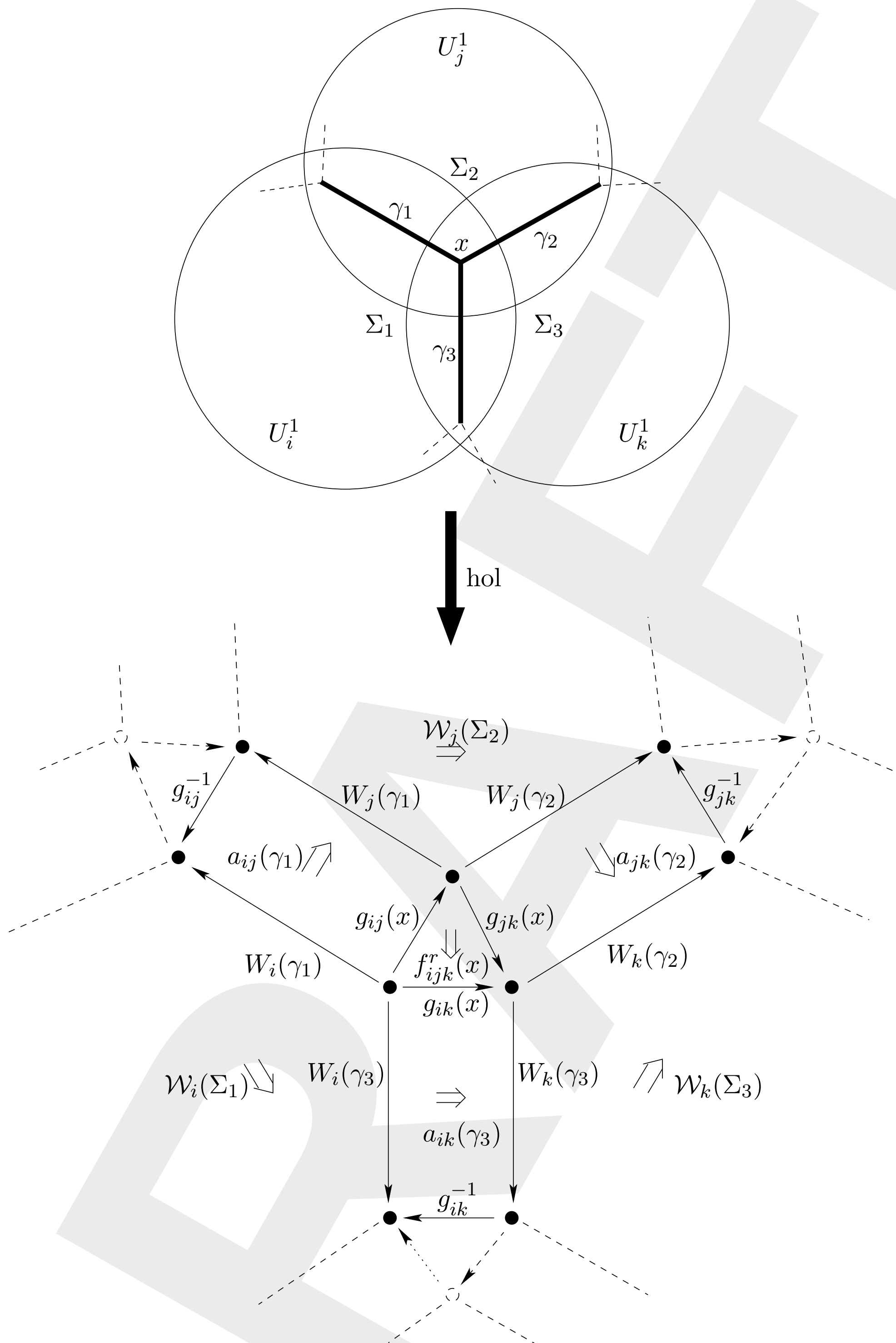
2.3 Global 2-Holonomy

Given any closed surface in base space whose 2-holonomy is to be computed we can triangularize it such a way that each face comes to lie in an element of the cover, each edge in a double overlap and each vertex in a triple overlap. We can always assume the graph of the triangularization to be trivalent. (If it is not we replace the problematic vertices by small circles of edges.)

The task is to assign 2-group elements to faces, edges and vertices of the triangularization such that the result of gluing them all together is independent of the choice of gauge (trivialization) as well as of the choice of cover and the choice of triangularization. For now we restrict attention on independence of the gauge choice.

It is clear that local 2-holonomies $\mathcal{W}_i(\Sigma)$ must be assigned to faces Σ. The only candidate 2-group elements to be assigned to edges γ are $a_{ij}(\gamma)$ and the only candidate 2-group elements to be assigned to vertices x are $f_{ijk}(x)$.

There is only one way to glue all these pieces consistently:

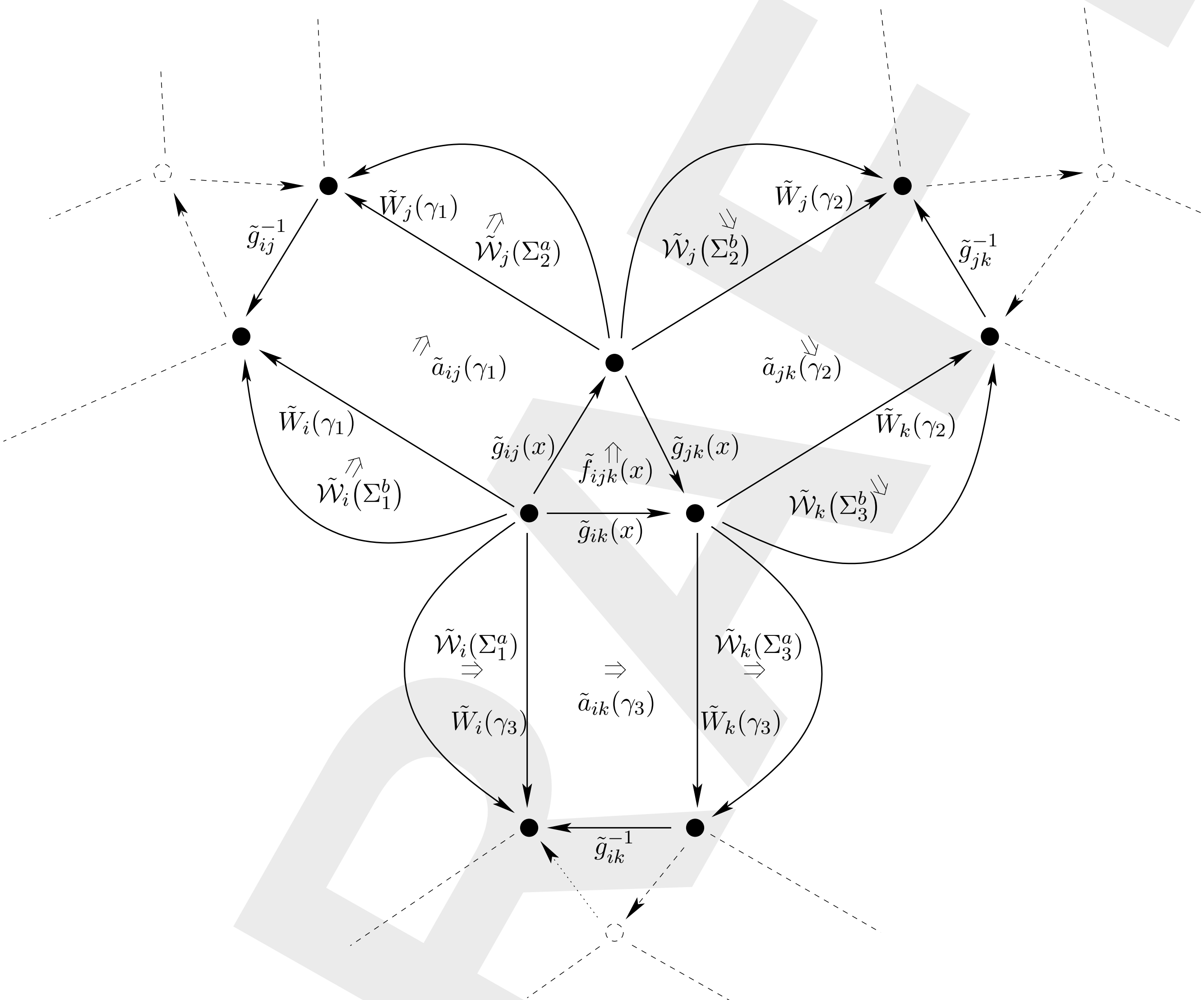


The result of multiplying all these 2-group elements together is the (global) 2-holonomy of our surface.

A glance at the definition of the global surface holonomy for an abelian gerbe with connection and curving shows that this is reproduced by the above diagram.

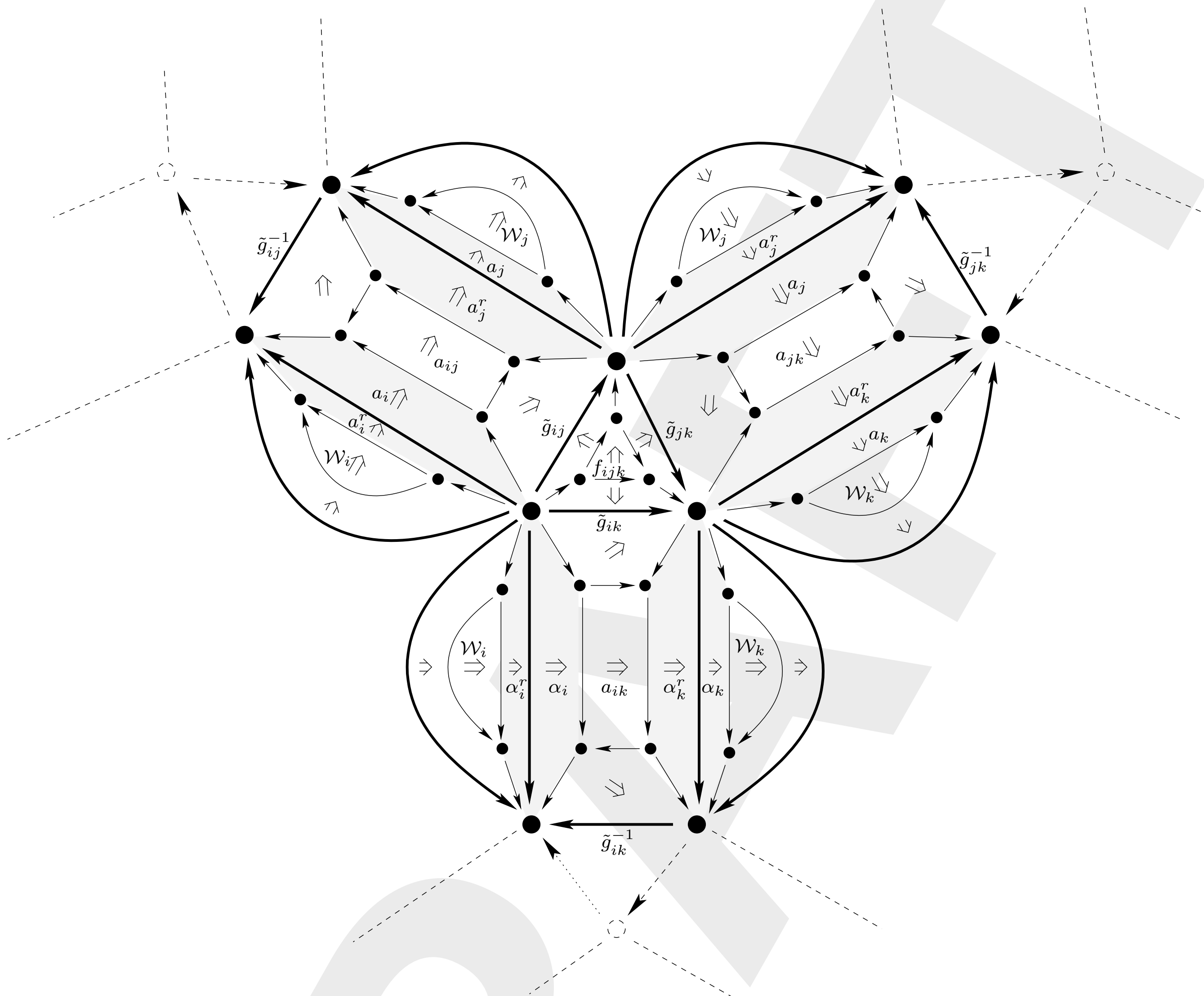
2.4 Gauge Invariance of Global 2-Holonomy

Fix the gauge $\tilde{G}$ and let the surface holonomy in the vicinity of some vertex x be given by



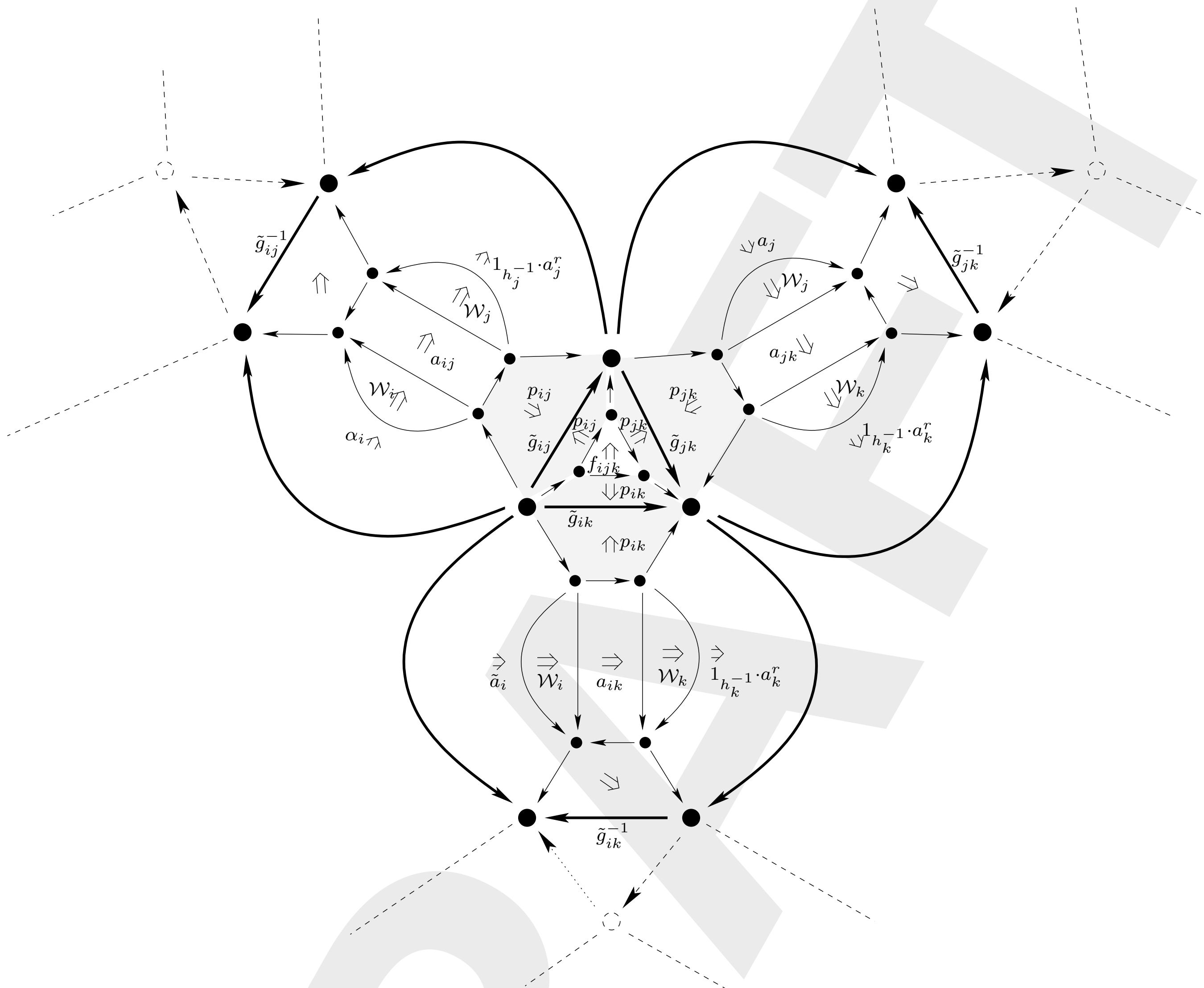
In this diagram the full local surface holonomies $\tilde{\mathcal{W}}_i(\Sigma_1)$, $\tilde{\mathcal{W}}_k(\Sigma_2)$, $\tilde{\mathcal{W}}_j(\Sigma_3)$ are depicted only in terms of two surface sub-elements $\Sigma_i^a, \Sigma_i^b \subset \Sigma_i$ etc., respectively, adjacent to the edges meeting at the given vertex.

Now we insert into this diagram the equalities discussed in §2.2 (p.4), which express the diagarams in the gauge $\tilde{G}$ in terms of those in the gauge G:

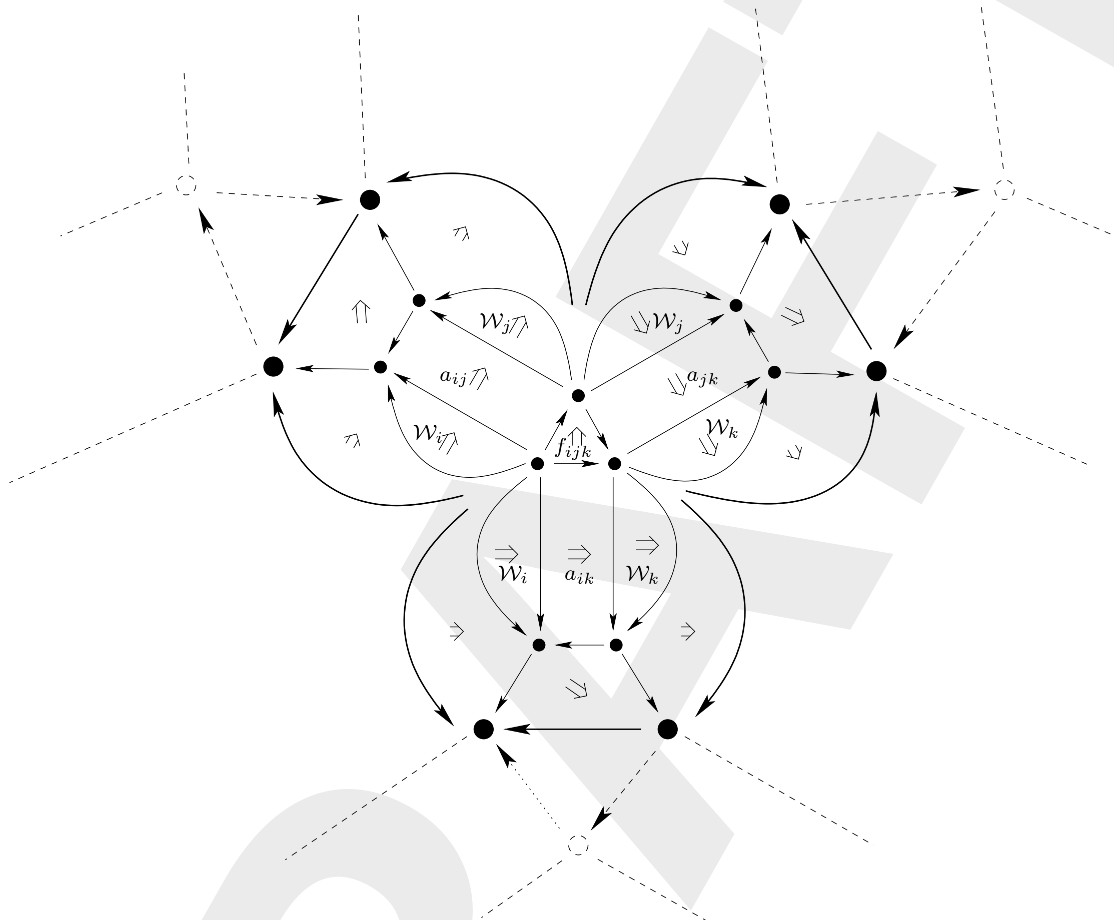


The result is that several 2-group elements are now adjacent which mutually cancel to unity. First of all we can cancel a_{ij} against a_{ij}^r and analogously for jk and ij. The respective identity 2-morphisms have been shaded in the diagram.

After removing them we are left with the following diagram



We have also reversed the direction of some edges by whiskering, so that we can now cancel p_{ij} against p_{ij}^r. When the shaded identity 2-morphisms are removed one obtains the following diagram



In the center of this diagram the surface holonomy in the gauge G has appeared. It is surrounded by 2-conjugations which cancel against the contributions from the other vertices.

This proves that the gobal 2-holonomy of a closed surface receives under gauge transformations at most a 2-conjugation with respect to the chosen source and target edge.

2.5 Trace of the global 2-Holonomy

But for strict structure 2-group this 2-holonomy is labeled by $h \in \ker(t) \subset H$ and hence automatically invariant under conjugation in H (see prop 3.1).

Full gauge invariance is hence obtained by tracing the result suitably so that the action of G on H drops out. Since it is only $G/\mathrm{im}(t)$ which acts nontrivially on $\ker(t)$ (by the same prop 3.1) it is sufficient to find a notion of trace that gets rid of this action.

2.6 Derivation of 2-Gauge Transformations from local Trivializations

The above transformation laws follow from a local 2-trivialization of a 2-bundle:

The 2-trivializations

$$E_{|U_i} \xrightarrow{t_i} U_i \times \mathcal{G}$$

in a $\mathcal{G}$-2-bundle E give rise to 2-transition 2-maps

$$U_{ij} \times \mathcal{G} \stackrel{\bar{t}_j \circ t_i}{\longrightarrow} U_{ij} \times \mathcal{G}$$

$$\left[\begin{pmatrix}(x,i,j)\\ \gamma \big\downarrow \\ (y,i,j)\end{pmatrix} \times \left(\bullet \overset{g_1}{\underset{g_2}{\Downarrow f}} \bullet\right)\right] \mapsto \left[\begin{pmatrix}(x,i,j)\\ \gamma \big\downarrow \\ (y,i,j)\end{pmatrix} \times \left(\bullet \overset{g^1_{ij}(x)}{\underset{t\left(g^2_{ij}(\gamma)\right)g^1_{ij}(x)}{\Downarrow g_{ij}(\gamma)}} \bullet \overset{g_1}{\underset{g_2}{\Downarrow f}} \bullet\right)\right]$$

that by definition act by left multiplication on the $\mathcal{G}$-factor and can be assumed to leave the U_{ij} factor alone.

This assumption/requirement still needs to be discussed in detail.

A natural transformation

$$\begin{pmatrix}(x,i,j)\\ \gamma \big\downarrow \\ (y,i,j)\end{pmatrix} \times \left(\bullet \overset{g^1_{ij}(x)}{\underset{t\left(g^2_{ij}(\gamma)\right)g^1_{ij}(x)}{\Downarrow g_{ij}(\gamma)}} \bullet \overset{g_1}{\underset{g_2}{\Downarrow f}} \bullet\right) \quad \begin{matrix}\xrightarrow{\tau((x,i,j))}\\ \\ \xrightarrow[\tau((y,i,j))]{}\end{matrix} \quad \begin{pmatrix}(x,i,j)\\ \gamma \big\downarrow \\ (y,i,j)\end{pmatrix} \times \left(\bullet \overset{\tilde{g}^1_{ij}(x)}{\underset{t\left(\tilde{g}^2_{ij}(\gamma)\right)\tilde{g}^1_{ij}(x)}{\Downarrow \tilde{g}_{ij}(\gamma)}} \bullet \overset{g_1}{\underset{g_2}{\Downarrow f}} \bullet\right)$$

between two such 2-transition 2-maps g_{ij} and $\tilde{g}_{ij}$ both mapping $U_{ij} \times \mathcal{G} \to U_{ij} \times \mathcal{G}$ is hence given by a natural transformation

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{g} & \mathcal{G} \\ & \Big\Downarrow & \\ U^{[2]} & \xrightarrow{\tilde{g}} & \mathcal{G}\,. \end{array}$$

In particular, a change of local trivializations defined by maps

$$\begin{array}{ccc} U & \xrightarrow{h} & \mathcal{G} \\ (\gamma_{x,y}, i, j) & \to & h_i(\gamma_{x,y}) \end{array}$$

gives rise to the **gauge transformation**

$$\begin{array}{ccc} U^{[2]} & \xrightarrow{\tilde{g}} & \mathcal{G} \\ & \Big\Downarrow p & \\ U^{[2]} & \xrightarrow{h_1 \cdot g \cdot h_2^{-1}} & \mathcal{G}\,. \end{array}$$

This implies that there is p_{ij} such that

$$p_{ij} \circ \tilde{g}_{ij} = \left(h_i \cdot g_{ij} \cdot h_j^{-1}\right) \circ p_{ij}\,.$$

2.7 Formulas

In this subsection some formulas describing the above diagrammatic reasoning are derived. It is shown that in the abelian case our global 2-holonomy reproduces that known from abelian gerbes.

Hence first recall some basic facts about Deligne hyper cohomology.

2.7.1 Deligne Hypercohomology

Denote by

$$D = \mathbf{d} + (-1)^{\hat{N}}\delta$$

the Deligne coboundary operator which is the sum of the deRham exterior derivative with the Čech coboundary operator times the deRham degree.

It acts on Deligne -1,0,1-chains as follows:

$$\begin{aligned}
D\left[\ln h_i\right] &= \left[(\ln h_j - \ln h_i)\,,\right.\\
&\qquad \left.(\mathbf{d}\ln h_i)\right]\\
D\left[\ln g_{ij}, A_i\right] &= \left[(\ln g_{jk} - \ln g_{ik} + \ln g_{ij})\,,\right.\\
&\qquad (\mathbf{d}\ln g_{ij} - A_j + A_i)\,,\\
&\qquad \left.(\mathbf{d}A_i)\right]\\
D\left[\ln f_{ijk}, a_{ij}, B_i\right] &= \left[(\ln f_{jkl} - \ln f_{ikl} + \ln f_{ijl} - \ln f_{ijk})\,,\right.\\
&\qquad (\mathbf{d}f_{ijk} - a_{jk} + a_{ik} - a_{ij})\,,\\
&\qquad (\mathbf{d}a_{ij} + B_j - B_i)\,,\\
&\qquad \left.(\mathbf{d}B_i)\right]
\end{aligned} \tag{2.2}$$

The equivalence class of an abelian p-gerbe is the p-th Deligne cohomology class, i.e. the equivalence class of Deligne chains

$$[\ldots]^{[p]} + D[\ldots]^{[p-1]}$$

with

$$D[\ldots]^{[p]} = [\cdots, ((p+1)\text{-form curvature})_i]$$

- a -1-gerbe is simply a **function**

  $$D[h_i] = [0, \mathbf{d}\ln h_i] \Rightarrow h_i = h$$

  the curvature 1-form of the -1-gerbe is the gradient of that function.

- a 0-gerbe is an **abelian bundle**, adding an exact Deligne chain corresponds to a gauge transformation in the local trivialization of the bundle (the gauge transformation corresponds to a “twisted” -1-gerbe, i.e. an ordinary function)

- an abelian 1-gerbe is an **abelian 1-gerbe**, gauge transformations in 1-gerbes come from “twisted bundles”

- and so on

2.7.2 Gauge Transformations of the Transition Function

The gauge transformation for the transition function is

$$1_{\tilde{g}_{ij}} = p^r_{ij} \circ 1_{h_i g_{ij} h_j^{-1}} \circ p_{ij}$$

This induces a gauge transformation on the f_{ijk}: In the new gauge we have

$$1_{\tilde{g}_{ij}\tilde{g}_{jk}} = \tilde{f}^r_{ijk} \circ 1_{\tilde{g}_{ik}} \circ \tilde{f}_{ijk} \,.$$

Expressed in terms of the original gauge this says that

$$\begin{aligned}
& \left(p^r_{ij} \circ 1_{h_i g_{ij} h_j^{-1}} \circ p_{ij}\right) \cdot \left(p^r_{jk} \circ 1_{h_j g_{jk} h_k^{-1}} \circ p_{jk}\right) = \tilde{f}^r_{ijk} \circ \left(p^r_{ik} \circ 1_{h_i g_{ik} h_k^{-1}} \circ p_{ik}\right) \circ \tilde{f}_{ijk} \\
\Leftrightarrow\ & (p^r_{ij} \cdot p^r_{jk}) \circ 1_{h_i g_{ij} g_{jk} h_k^{-1}} \circ (p_{ij} \cdot p_{jk}) = \tilde{f}^r_{ijk} \circ p^r_{ik} \circ 1_{h_i g_{ik} h_k^{-1}} \circ p_{ik} \circ \tilde{f}_{ijk} \\
\Leftrightarrow\ & 1_{g_{ij} g_{jk}} = 1_{h_i^{-1}} \cdot \left((p_{ij} \cdot p_{jk}) \circ \tilde{f}^r_{ijk} \circ p^r_{ik} \circ 1_{h_i g_{ik} h_k^{-1}} \circ p_{ik} \circ \tilde{f}_{ijk} \circ (p^r_{ij} \cdot p^r_{jk})\right) \cdot 1_{h_k} \\
& = \left(1_{h_i^{-1}} \cdot \left((p_{ij} \cdot p_{jk}) \circ \tilde{f}^r_{ijk} \circ p^r_{ik}\right) \cdot 1_{h_k}\right) \circ 1_{g_{ik}} \circ \left(1_{h_i^{-1}} \cdot \left(p_{ik} \circ \tilde{f}_{ijk} \circ (p^r_{ij} \cdot p^r_{jk})\right) \cdot 1_{h_k}\right) \,.
\end{aligned}$$

Hence (up to a twist)

$$\begin{aligned}
& f_{ijk} = 1_{h_i^{-1}} \cdot \left(p_{ik} \circ \tilde{f}_{ijk} \circ (p_{ij} \cdot p_{jk})^r\right) \cdot 1_{h_k} \\
\Leftrightarrow\ & \tilde{f}_{ijk} = p^r_{ik} \circ \left(1_{h_i} \cdot f_{ijk} \cdot 1_{h_k^{-1}}\right) \circ (p_{ij} \cdot p_{jk})
\end{aligned}$$

Denote as before 2-group elements by their G and H components as

$$\begin{aligned}
h_i &\equiv (h_i, 1) \\
g_{ij} &\equiv (g_{ij}, 1) \\
p_{ij} &\equiv (p^1_{ij}, p^2_{ij}) \\
f_{ijk} &\equiv (f^1_{ijk}, f^2_{ijk}) \,.
\end{aligned}$$

Then at the point level (2.3) says that

$$\tilde{g}_{ij} = t\big(p^2_{ij}\big)\, h_i\, g_{ij}\, h_j^{-1} \,.$$

This already implies the gauge transformation of $t\big(f^2\big)$:

$$\begin{aligned}
& \tilde{g}_{ij}\tilde{g}_{jk} = t\Big(\tilde{f}^2_{ijk}\Big)\, \tilde{g}_{ik} \\
\Leftrightarrow\ & t\big(p^2_{ij}\big)\, h_i g_{ij} h_j^{-1} t\big(p^2_{jk}\big)\, h_j g_{jk} h_k^{-1} = t\Big(\tilde{f}^2_{ijk}\Big)\, t\big(p^2_{ik}\big)\, h_i g_{ik} h_k^{-1} \,.
\end{aligned}$$

This implies that

$$t\big(f^2_{ijk}\big) = g_{ij} h_j^{-1} t\big(p^2_{jk}\big)^{-1} h_j g_{ij}^{-1} h_i^{-1} t\Big((p^2_{ij})^{-1} \tilde{f}^2_{ijk} p^2_{ik}\Big)\, h_i \tag{2.3}$$

$$= t\Big(\alpha\Big(g_{ij} h_j^{-1}\Big)\big((p^2_{jk})^{-1}\big)\, \alpha\big(h_i^{-1}\big)\Big((p^2_{ij})^{-1} \tilde{f}^2_{ijk} p^2_{ik}\Big)\Big) \,. \tag{2.4}$$

Up to a factor in ker(t) (kind of a twist again) this implies that the transformed $\tilde{f}^2_{ijk}$ is

$$\tilde{f}^2_{ijk} = p_{ij}\alpha\Big(h_i g_{ij} h_j^{-1}\Big)\big(p^2_{jk}\big)\,\alpha(h_i)\big(f^2_{ijk}\big)\,(p^2_{ik})^{-1}\,, \tag{2.5}$$

which is what also follows from (2.3).

In the abelian case this reduces to

$$\ln \tilde{f}^2_{ijk} = \ln f^2_{ijk} + \ln p^2_{jk} - \ln p^2_{ik} + \ln p^2_{ij}\,, \tag{2.6}$$

which is indeed the transition function part of the gauge transformation in an abelian gerbe (2.2).

2.7.3 Gauge Transformations of the 2-Connection

The 2-gauge transformed transition law is

$$\tilde{\mathcal{W}}_i = \tilde{\alpha}_{ij} \circ \left(1_{\tilde{g}_{ij}} \cdot \tilde{\mathcal{W}}_j \cdot 1_{\tilde{g}_{ij}^{-1}}\right) \circ \tilde{\alpha}^r_{ij}\,.$$

In terms of the original gauge this says (for $p_{ij} = 1_{g_{ij}}$) that

$$\begin{aligned}
&\alpha_i \circ \left(1_{h_i} \cdot \mathcal{W}_i \cdot 1_{h_i^{-1}}\right) \circ \alpha^r_i = \tilde{\alpha}_{ij} \circ \left(1_{h_i g_{ij} h_j^{-1}} \cdot \left(\alpha_j \circ \left(1_{h_j} \cdot \mathcal{W}_j \cdot 1_{h_j^{-1}}\right) \circ \alpha^r_j\right) \cdot 1_{(h_i g_{ij} h_j^{-1})^{-1}}\right) \circ \tilde{\alpha}^r_{ij} \\
\Leftrightarrow\ & 1_{h_i} \cdot \mathcal{W}_i \cdot 1_{h_i^{-1}} = (\alpha^r_i \circ \tilde{\alpha}_{ij}) \circ \left(1_{h_i g_{ij} h_j^{-1}} \cdot \left(\alpha_j \circ \left(1_{h_j} \cdot \mathcal{W}_j \cdot 1_{h_j^{-1}}\right) \circ \alpha^r_j\right) \cdot 1_{(h_i g_{ij} h_j^{-1})^{-1}}\right) \circ (\tilde{\alpha}^r_{ij} \circ \alpha_i) \\
\Leftrightarrow\ & \mathcal{W}_i = 1_{h_i^{-1}} \cdot \left((\alpha^r_i \circ \tilde{\alpha}_{ij}) \circ \left(1_{h_i g_{ij} h_j^{-1}} \cdot \left(\alpha_j \circ \left(1_{h_j} \cdot \mathcal{W}_j \cdot 1_{h_j^{-1}}\right) \circ \alpha^r_j\right) \cdot 1_{(h_i g_{ij} h_j^{-1})^{-1}}\right) \circ (\tilde{\alpha}^r_{ij} \circ \alpha_i)\right) \cdot 1_{h_i} \\
&= \left(1_{h_i^{-1}} \cdot (\alpha^r_i \circ \tilde{\alpha}_{ij}) \cdot 1_{h_i}\right) \circ \left(1_{g_{ij} h_j^{-1}} \cdot \left(\alpha_j \circ \left(1_{h_j} \cdot \mathcal{W}_j \cdot 1_{h_j^{-1}}\right) \circ \alpha^r_j\right) \cdot 1_{(g_{ij} h_j^{-1})^{-1}}\right) \circ \left(1_{h_i^{-1}} \cdot (\tilde{\alpha}^r_{ij} \circ \alpha_i) \cdot 1_{h_i}\right) \\
&= \left(1_{h_i^{-1}} \cdot (\alpha^r_i \circ \tilde{\alpha}_{ij}) \cdot 1_{h_i}\right) \circ \\
&\circ \left(1_{g_{ij} h_j^{-1}} \cdot \alpha_j \cdot 1_{(g_{ij} h_j^{-1})^{-1}}\right) \circ \left(1_{g_{ij}} \cdot \mathcal{W}_j \cdot 1_{g_{ij}^{-1}}\right) \circ \left(1_{g_{ij} h_j^{-1}} \cdot \alpha^r_j \cdot 1_{(g_{ij} h_j^{-1})^{-1}}\right) \circ \\
&\circ \left(1_{h_i^{-1}} \cdot (\tilde{\alpha}^r_{ij} \circ \alpha_i) \cdot 1_{h_i}\right)\,.
\end{aligned}$$

By comparison with the transition law

$$\mathcal{W}_i = \alpha_{ij} \circ \left(1_{g_{ij}} \cdot \mathcal{W}_j \cdot 1_{g_{ij}^{-1}}\right) \circ \alpha^r_{ij}\,.$$

in the original gauge, this implies that

$$a_{ij} = \left(1_{h_i^{-1}} \cdot (\alpha^r_i \circ \tilde{\alpha}_{ij}) \cdot 1_{h_i}\right) \circ \left(1_{g_{ij} h_j^{-1}} \cdot \alpha_j \cdot 1_{(g_{ij} h_j^{-1})^{-1}}\right)\,.$$

Solving for $\tilde{a}_{ij}$ we obtain the gauge transformation law for a_{ij}

$$\begin{aligned}
\tilde{a}_{ij} &= \alpha_i \circ \left(1_{h_i} \cdot \left(a_{ij} \circ \left(1_{g_{ij} h_j^{-1}} \cdot \alpha^r_j \cdot 1_{(g_{ij} h_j^{-1})^{-1}}\right)\right) \cdot 1_{h_i^{-1}}\right) \\
&= \alpha_i \circ \left(1_{h_i} \cdot a_{ij} \cdot 1_{h_i^{-1}}\right) \circ \left(1_{\tilde{g}_{ij}} \cdot \alpha^r_j \cdot 1_{\tilde{g}_{ij}^{-1}}\right)
\end{aligned}$$

In order to evaluate this in terms of the $(1,2)$-form (A, B) it is more convenient to proceed as follows:

Since a 2-transition for the connection is really nothing but a gauge transformation we can simply copy the relevant formulas to find that the above change in local 2-trivialization gives a new $(1,2)$ form $(\tilde{A}_i, \tilde{B}_i)$ with

$$\begin{aligned} \tilde{A}_i &= h_i(\mathbf{d} + A_i)h_i^{-1} - dt(\alpha_i) \\ \tilde{B}_i &= \alpha(h_i)(B_i) - \mathbf{d}_{A_i}\alpha_i - \alpha_i \wedge \alpha_i \,. \end{aligned} \tag{2.7}$$

This implies a gauge transformation for the natural transformation encoded in a_{ij}:

$$\begin{aligned} & \tilde{A}_i + dt(\tilde{a}_{ij}) = \tilde{g}_{ij}(\tilde{A}_j + \mathbf{d})\tilde{g}_{ij}^{-1} \\ \Leftrightarrow\ & h_i(A_i + \mathbf{d})h_i^{-1} - dt(\alpha_i) + dt(\tilde{a}_{ij}) \\ & = t(p_{ij})\, h_i g_{ij} h_j^{-1}(h_j A_j h_j^{-1} - dt(\alpha_j) + (h_j \mathbf{d} h_j^{-1}) + \mathbf{d}) h_j g_{ij}^{-1} h_i^{-1} t(p_{ij})^{-1} \,. \end{aligned} \tag{2.8}$$

It follows that the gauge transformed $\tilde{a}_{ij}$ is

$$\begin{aligned} \tilde{a}_{ij} &= a_{ij} + \alpha_i - \alpha\Big(t(p_{ij})\, h_i g_{ij} h_j^{-1}\Big)(\alpha_j) + p_{ij}\mathbf{d}p_{ij}^{-1} \\ &\quad + \dots \end{aligned} \tag{2.9}$$

where we omit a mess of terms that vanish in the abelian case.

Hence for abelian 2-bundles we get

$$\tilde{a}_{ij} = a_{ij} + \alpha_i - \alpha_j - \mathbf{d}\ln p_{ij} \,. \tag{2.10}$$

This again coincides with the result from Deligne cohomology (up to signs maybe that need to be made consistent) (2.2).

The abelian gauge transformation rule for B_i also follows immediately.

Hence changes of local trivializations in $\mathcal{G}$-2-bundles do reproduce the well known gauge transformations in cocylces of abelian gerbes. But they are much more general. I am not sure if the full nonabelian gauge transformation laws sketched above have been discussed before in the context of nonabelian gerbes. There is more in Breen&Messing than meets the eye...

3. 2-Curvature

The following lists some observations concerning the curvature in a $\mathcal{G}$-2-bundle.

Proposition 3.1 *In a crossed module* $\mathrm{im}(t)$ *acts trivially on* $\ker(t)$*. Equivalently, in a differential crossed module* $\mathrm{im}(dt)$ *acts trivially on* $\ker(dt)$*.*

Proof. This is a consequence of the property

$$\begin{aligned} \alpha(t(h_1))(h_2) &= h_1 h_2 h_1^{-1} \\ d\alpha(dt(S_1))(S_2) &= [S_1, S_2] \end{aligned} \tag{3.1}$$

of a crossed module, with $h_i \in H$ and $S_i \in \mathfrak{h}$:

Let $h \in H$ and $k \in \ker(t) \subset H$. Then

$$\begin{aligned} \alpha(t(h))(k) &= hkh^{-1} \\ &= kk^{-1}hkh^{-1} \\ &= k\,\underbrace{\alpha\big(t\big(k^{-1}\big)\big)(h)}_{=h}\,h^{-1} \\ &= k\,. \end{aligned}$$

Similarly for differential crossed modules with $S \in \mathfrak{h}$ and $S_0 \in \ker(dt) \subset \mathfrak{h}$:

$$\begin{aligned} d\alpha(dt(S))(S_0) &= [S, S_0] \\ &= -\,[S_0, S] \\ &= -\,\underbrace{d\alpha(dt(S_0))}_{=0}(S) \\ &= 0\,. \end{aligned} \tag{3.2}$$

□

Proposition 3.2 *The vanishing of the fake curvature implies that*

$$F_A \wedge B = 0\,,$$

which is shorthand for

$$(F_A^a \wedge B^b)\; d\alpha(T_a)(S_b) = 0\,.$$

Proof. Use $F_A = -dt(B)$ to get

$$\begin{aligned} (F_A^a \wedge B^b)\; d\alpha(T_a)(S_b) &= -(B^a \wedge B^b)\; d\alpha(dt(S_a))(S_b) \\ &= -(B^a \wedge B^b)\;\; [S_a, S_b] \\ &= 0\,. \end{aligned}$$

This vanishes because $B^a \wedge B^b = B^b \wedge B^a$ (since B is a 2-form) while $[S_a, S_b] = -[S_b, S_a]$.
□

Remark: This is equivalent to the Bianchi-identity on path space:
The curvature on path space for vanishing fake curvature is (corollary 2.2 in [2])

$$\begin{aligned}\mathcal{F}_{\mathcal{A}} &= \left(\mathbf{d} + \oint_A (B)\right)^2 \\ &\equiv -\oint_A (\mathbf{d}_A B)\;.\end{aligned}$$

The Bianchi-Identity says that

$$\begin{aligned}0 &= \mathbf{d}\oint_A (H) + \underbrace{\oint_A (B)\wedge\oint_A (H) + \oint_A (H)\wedge\oint_A (B)}_{=0 \text{ by prop. } 3.1} \\ &= \mathbf{d}\oint_A (H) \\ &= -\oint_A (\mathbf{d}_A H) - \oint_A (d\alpha(T_a)(H)\,,F^a) \\ &= -\oint_A (\mathbf{d}_A H) + \oint_A (d\alpha(T_a)(H)\,,dt(B)^a) \\ &= \oint_A (\mathbf{d}_A H)\;.\end{aligned}$$

This means that the vanishing of the fake curvature ensures that the 3-form "field strength" is still covariantly closed. This again ensures that self-duality of the field strength, i.e.

$$H = \pm \star H$$

is sufficient to imply equations of motion of the form

$$\mathbf{d}_A \star H = 0\,.$$

In the abelian case this ensures that the 6-dimensional self-dual theory compactifies to an ordinary gauge theory. Vanishing fake curvature also ensures that this is gauge invariant.

This observation arose in a discussion with Jens Fjelstad.

4. More on Path Space Calculus

4.0.4 Exterior coderivative and Yang-Mills equations

In the presence of a metric structure in ordinary differential geometry the exterior derivative is accompanied by its adjoint with respect to the Hodge inner product, the (exterior) coderivative. In the case where target space is equipped with a metric a corresponding metric is induced on path space [3] and we can consider the formal Hodge adjoint $\mathbf{d}^\dagger$ of the exterior derivative on path space which we shall write as

$$\mathbf{d}^\dagger = -dX^{\dagger(\mu,\sigma)}\hat{\nabla}_{(\mu,\sigma)}\,, \tag{4.1}$$

where the dagger is suggestive but purely formal notation motivated [3] by the respective expressions in finite dimensional differential geometry (as for instance described in the appendix of [4]), so that $dX^\dagger$ must be thought of as inner multiplication $dX \rightharpoonup$, to be made precise below. In the context of the tensile string (i.e. the string with non-vanishing tension) this operator $\mathbf{d}^\dagger$ is one part of the worldsheet supercharge [3], which is a well defined operator on the Hilbert space of the string. For the tensionless string or for purposes of pure path space differential geometry we will now show how to make sense of the formal expression (4.1) acting on the space of (formal power series of) pull-back forms.

To that end, first note that for flat target space we formally have

$$\begin{aligned}
\mathbf{d}^\dagger \oint(\omega_1,\cdots,\omega_n) &= -dX^\dagger\cdot\partial \int\limits_{\sigma_1\leq\cdots\leq\sigma_n} X'\cdot\omega_1(\sigma_1)\cdots X'\cdot\omega_n(\sigma_n)\\
&= \sum_{k=1}^{n}\int\limits_{\sigma_1\leq\cdots\leq\sigma_n} -dX^{\dagger\prime\mu}(\sigma_k)\,X'\cdot\omega_1(\sigma_1)\cdots(\omega_k)_\mu(\sigma_k)\cdots X'\cdot\omega_n(\sigma_n)\\
&\quad+\sum_{k=1}^{n}\int\limits_{\sigma_1\leq\cdots\leq\sigma_n} -dX^{\dagger\mu}(\sigma_k)\,X'\cdot\omega_1(\sigma_1)\cdots\cdot X'\cdot(\partial_\mu\omega_k)(\sigma_k)\cdots X'\cdot\omega_n(\sigma_n)\;.
\end{aligned} \tag{4.2}$$

At this point a problem arises due to $dX_\mu(\sigma) \rightharpoonup dX^\nu(\sigma') = \delta^\nu_\mu\delta(\sigma-\sigma')$ which makes $dX^{\mu\dagger}(\sigma_k)\,\omega(\sigma_k)$ ill defined. This problem can be traced back to the fact that acting with $\mathbf{d}^\dagger$ on differential forms in a finite dimensional context produces derivatives of the determinant of the metric which is not well defined (formally infinite) on path space. But it is immediate how to regularize $\mathbf{d}^\dagger$ when acting on pull-back forms:

Imagine for the moment a discretization of the paths in path space so that discretized paths are maps from $\{0,\epsilon,2\epsilon,3\epsilon,\cdots,1\}$ into target space with $\epsilon = 1/N$ for some large natural number N. With a suitable lattice version of integrals over the path parameter and derivatives with respect to this parameter the discretization of the above expression is asymptotic to

$$\begin{aligned}
&\to \frac{1}{\epsilon}\sum_k(-1)^{\sum\limits_{i<k}p_i}\int\limits_{\sigma_1\leq\cdots\leq\sigma_n} X'\cdot\omega_1,\cdots,X'^\nu(\omega_k)_{\nu\mu}(\omega_{k+1})^\mu-(-1)^{p_k}(\omega_k)^\mu X'^\nu(\omega_{k+1})_{\nu\mu},\cdots,X'\cdot\omega_n\\
&\quad-\frac{1}{\epsilon}\sum_k(-1)^{\sum\limits_{i<k}p_i}\oint\Big(\omega_1,\cdots,\mathbf{d}^\dagger\omega_k,\cdots,\omega_n\Big)\\
&.
\end{aligned} \tag{4.3}$$

Comparison with (**??**) shows that this is a perfectly sensible looking expression except for the prefactor $1/\epsilon$, which diverges in the continuum limit $N = 1/\epsilon \to \infty$. It therefore makes sense to *define* the action of $\mathbf{d}^\dagger$ on pull-back forms to be given by that formula with the prefactor replaced by unity. This gives the desired definition

$$\begin{aligned}
&\mathbf{d}^\dagger \oint (\omega_1, \cdots, \omega_n) \\
\equiv &-\sum_k (-1)^{\sum\limits_{i<k} p_i} \oint \Big(\omega_1, \cdots, d^\dagger \omega_k, \cdots, \omega_n\Big) \\
&+\sum_k (-1)^{\sum\limits_{i<k} p_i} \int\limits_{\sigma_1 \leq \cdots \leq \sigma_n} X'\cdot\omega_1, \cdots, X'^\nu (\omega_k)_{\nu\mu} (\omega_{k+1})^\mu - (-1)^{p_k} (\omega_k)^\mu X'^\nu (\omega_{k+1})_{\nu\mu}, \cdots, X'\cdot\omega_n \,.
\end{aligned} \tag{4.4}$$

We have shown the derivation of this expression for flat target space but it directly generalizes to arbitrary metrics.

Unfortunately this expression is not manifestly a formal power series of pull-back forms. However for special cases it does become one. In particular for $\left\{\omega_i^{(1)}\right\}_{i=1^n}$ any family of 1-forms and $\omega^{(2)}$ any 2-form on target space we have

$$\begin{aligned}
\mathbf{d}^\dagger \oint \Big(\omega_1^{(1)}, \dots, \omega_i^{(1)}, \omega^{(2)}, \omega_{i+1}^{(1)}, \dots, \omega_n^{(1)}\Big) = &- \oint \Big(\omega_1^{(1)}, \dots, \omega_i^{(1)}, d^\dagger \omega^{(2)}, \omega_{i+1}^{(1)}, \dots, \omega_n^{(1)}\Big) \\
&- \oint \Big(\omega_1^{(1)}, \dots, \omega_i^{(1)}, \omega_\mu^{(2)} \omega^{(1)}{}_{i+1}^{\mu}, \omega_{i+2}^{(1)} \dots, \omega_n^{(1)}\Big) \\
&+ \oint \Big(\omega_1^{(1)}, \dots, \omega_{i-1}^{(1)} \omega^{(1)}{}_i^{\mu} \omega_\mu^{(2)}, \omega_{i+1}^{(1)}, \dots, \omega_n^{(1)}\Big) \,.
\end{aligned} \tag{4.5}$$

The point of this becomes more obvious for Wilson line-like pull-back forms. For A any 1-form and B any 2-form on target space it follows that

$$\mathbf{d}^\dagger \sum_{n,m=0}^{\infty} \oint (\underbrace{iA, \dots, iA}_{n \text{ times}}, B, \underbrace{iA, \dots, iA}_{m \text{ times}}) = \sum_{n,m=0}^{\infty} \oint (\underbrace{iA, \dots, iA}_{n \text{ times}}, -\mathbf{d}^\dagger B + \mathrm{ad}(iA_\mu)(B^\mu)\,, \underbrace{iA, \dots, iA}_{m \text{ times}}) \,. \tag{4.6}$$

This is of interest in the study of loop space formulations of the Yang-Mills equations:

A pull-back form of central importance is the holonomy W_A of a target space 1-form A along the path

$$W_A = \sum_{n=0}^{\infty} \oint (iA^{a_1}, \cdots, iA^{a_n})\, T_{a_1} \cdots T_{a_n} \,, \tag{4.7}$$

for which we also write

$$\begin{aligned}
W_A &\equiv W_A(0,1) \\
&= W_A(0,\sigma)\, W_A(\sigma,1) \,, \qquad \forall\, \sigma \in (0,1) \,.
\end{aligned} \tag{4.8}$$

With (**??**) the exterior differential of this 0-form is

$$\begin{aligned}\mathbf{d}W_A &= \mathbf{d}\sum_{n=0}^{\infty}\oint (iA^{a_1},\ldots,iA^{a_n})\,T_{a_1}\cdots T_{a_n}\\ &= -\sum_{n,m=0}^{\infty}\oint (iA^{a_1},\ldots,iA^{a_n},iF_A^a,iA^{a_{n+1}},\ldots,iA^{a_{n+m}})\,T_{a_1}\cdots T_{a_n}T_aT_{a_{n+1}}\cdots T_{a_{n+m}}\\ &= -\int_0^1 d\sigma\; W_A(0,\sigma)\,iF_A(\sigma)\,W_A(\sigma,1)\;. \end{aligned} \tag{4.9}$$

Acting with $\mathbf{d}^\dagger$ on this result produces

$$\begin{aligned}\left\{\mathbf{d},\mathbf{d}^\dagger\right\}W_A &= \mathbf{d}^\dagger\mathbf{d}W_A\\ &= -\mathbf{d}^\dagger\int_0^1 d\sigma\; W_A(0,\sigma)\,iF_A(\sigma)\,W_A(\sigma,1)\\ &\overset{(4.6)}{=} -i\int_0^1 d\sigma\; W_A(0,\sigma)\left(-d^\dagger F + \mathrm{ad}(iA_\mu)(F^\mu)\right)(\sigma)\,W_A(\sigma,1)\;. \end{aligned}\tag{4.10}$$

A similar expression is obtained when we first premultiply (4.9) with W_A^{-1} to get the simple expression

$$\begin{aligned}W_A^{-1}(\mathbf{d}W_A) &= -\int_0^1 d\sigma\;(W_A(\sigma,1))^{-1}iF_A(\sigma)\,W_A(\sigma,1)\\ &= -\sum_{n=0}^{\infty}\oint (\mathrm{ad}(T_{a_n})\circ\cdots\circ\mathrm{ad}(T_{a_1})(iF_A)\,,-iA^{a_1},\ldots,-iA^{a_n})\\ &\equiv -i\oint (F_A^{W_A})\,. \end{aligned}\tag{4.11}$$

Then using (4.4) one gets

$$\begin{aligned}\mathbf{d}^\dagger\left(W_A^{-1}(\mathbf{d}W_A)\right) &= \sum_{n=0}^{\infty}\oint\Big(\mathrm{ad}(T_{a_n})\circ\cdots\circ\mathrm{ad}(T_{a_1})\Big(i(-\mathbf{d}^\dagger F_A + \mathrm{ad}(iA_\mu)((F_A)^\mu))\Big)\,,-iA^{a_1},\ldots,-iA^{a_n}\Big)\\ &= i\oint\Big((\partial_\mu F^\mu + \mathrm{ad}(iA_\mu)(F^\mu))^{W_A}\Big)\;. \end{aligned}\tag{4.12}$$

Related to this is the observation that when $W_A^{-1}(\mathbf{d}A)$ is regarded as a connection 1-form, its field strength vanishes. In our notation this is trivial, since $W_A^{-1}(\mathbf{d}W_A)$ is gauge equivalent to the trivial connection.

This reproduces the old observation by Polyakov [**?**, **?**] that (up to the formally infinite prefactor) the vanishing of the divergence of these objects is related (*cf.* [5]) to the classical equations of motion of Yang-Mills theory (in flat spacetime):

$$\partial_\mu F^{\mu\nu} + \mathrm{ad}(iA_\mu)(F^{\mu\nu}) = 0\,. \tag{4.13}$$

4.0.5 Hochschild operators

In [6] it was noted that equation (**??**) suggests that in differential calculus on pull-back forms on path space *graded multi-derivations* play an important role.

Clearly, if we consider the string of forms $(\omega_1, \ldots, \omega_n)$ as an n-fold associative abstract product of some unspecified sort and if we associate a grade

$$|\omega_i| \equiv p_i = \deg(\omega_i) - 1 \qquad (4.14)$$

with each factor (*cf.* (**??**)) then the first term in (**??**) can be thought of as coming from the application of the *unary* derivation ϕ_d of odd grade $|\phi_d| = 1$ defined by[1]

$$\phi_d(\omega_1, \ldots, \omega_n) \equiv \sum_{k=0}^{n-1} (-1)^{\sum_{i=1}^{k} |\phi_d||\omega_i|} (\omega_1, \ldots, \omega_k, -d\omega_{k+1}, \omega_{k+2}, \ldots, \omega_n) \qquad (4.15)$$

while the second term can be thought of as due to a *binary* derivation ϕ_M of grade $|\phi_M| = 1$

$$\phi_M(\omega_1, \ldots, \omega_n) \equiv \sum_{k=0}^{n-2} (-1)^{\sum_{i=1}^{k} |\phi_M||\omega_i|} (\omega_1, \ldots, \omega_k, M(\omega_{k+1}, \omega_{k+2}), \ldots, \omega_n)\,, \qquad (4.16)$$

where the binary map M is, up to a sign, the (wedge) product operation

$$M(\omega_1, \omega_2) = (-1)^{|\omega_1|} \omega_1 \wedge \omega_2\,. \qquad (4.17)$$

For these two (multi-)derivations it so happens that their grade $|\phi|$ (defined by the signs in the above sums) is related to the grade of their image (defined by (4.14)) simply by

$$|\phi(\omega_1, \ldots, \omega_n)| = |\phi| + \sum_{k=1}^{n} |\omega_k|\,. \qquad (4.18)$$

An arbitrary graded multi-derivation has no reason to satisfy this relation. But those that do have the nice property that they form a closed algebra under the graded Lie bracket given by the graded commutator

$$[\phi_1, \phi_2] \equiv \phi_1 \circ \phi_2 - (-1)^{|\phi_1||\phi_2|}\, \phi_2 \circ \phi_1\,. \qquad (4.19)$$

A simple calculation shows that this bracket respects the grade

$$|[\phi_1, \phi_2]| = |\phi_1| + |\phi_2| \qquad (4.20)$$

and that if ϕ_1 is n_1-ary and ϕ_2 is n_2-ary the resulting derivation is $(n_1 + n_2 - 1)$-ary and given by

$$\begin{aligned} &[\phi_1, \phi_2](\omega_1, \ldots, \omega_{n_1+n_2-1}) \\ = &\sum_{r=0}^{n_1-1} (-1)^{\sum_{i=1}^{r} |\phi_2||\omega_i|} \phi_1(\omega_1, \ldots, \omega_r, \phi_2(\omega_{r+1}, \ldots, \omega_{r+n_2}), \omega_{r+n_2+1}, \ldots, \omega_{r+n_1+n_2-1}) \\ &-(-1)^{|\phi_1||\phi_2|} \sum_{r=0}^{n_2-1} (-1)^{\sum_{i=1}^{r} |\phi_1||\omega_i|} \phi_2(\omega_1, \ldots, \omega_r, \phi_1(\omega_{r+1}, \ldots, \omega_{r+n_1}), \omega_{r+n_1+1}, \ldots, \omega_{r+n_1+n_2-1})\,. \end{aligned} \qquad (4.21)$$

[1] We have some (inessential) signs different from those in [6].

As noted in [6], this operation is related to the so-called Gerstenhaber bracket between multilinear maps.

In order to see this first note that for the pull-back forms (**??**) the grade $|\phi|$ of the multi-derivation which satisfy (4.18) is the sum of the differential form degree they carry and their arity reduced by one, i.e.

$$|\phi| = \deg(\phi) + n_\phi - 1 \tag{4.22}$$

(because they remove n contractions with X' and add one.)

Next it is instructive to first restrict attention to the case where all form degrees of the ω as well as of the ϕ are *even*, which is the purely 'bosonic' case. In this case the grade $|\omega|$ of a form in (4.14) is *odd* and the grade $|\phi|$ of our multi-derivations is $n-1$.

So in this case (4.21) reduces to the ordinary *Gerstenhaber bracket*

$$\begin{aligned}
[\phi_1, \phi_2] &= \phi_1(\phi_2(\dots), \dots) - (-1)^{(n_1-1)(n_2-1)} \phi_2(\phi_1(\dots), \dots) \\
&\quad + \Big((-1)^{(n_2-1)} \phi_1(\cdot, \phi_2(\dots), \dots) - (-1)^{(n_1-1)((n_2-1)+1)} \phi_2(\cdot, \phi_1(\dots), \dots) \Big) \\
&\quad + \Big(\phi_1(\cdot, \cdot, \phi_2(\dots), \dots) - (-1)^{(n_1-1)(n_2-1)} \phi_2(\cdot, \cdot, \phi_1(\dots), \dots) \Big) \\
&\quad + \cdots ,
\end{aligned} \tag{4.23}$$

which reproduces all the familiar algebraic relations between maps: For instance let M be binary, μ unary and α 0-ary (all of even form degree), then (for all their arguments of even form degree, too) the above says that

$$\begin{aligned}
[\mu(\cdot),\alpha] &= \mu(\alpha) \\
[\mu(\cdot),\nu(\cdot)] &= \mu(\nu(\cdot)) - \nu(\mu(\cdot)) \\
[M(\cdot,\cdot),\alpha] &= M(\alpha,\cdot) - M(\cdot,\alpha) \\
[M(\cdot,\cdot),\mu(\cdot)] &= M(\mu(\cdot)\,,\cdot) - M(\cdot,\mu(\cdot)) - \mu(M(\cdot,\cdot)) \\
[M(\cdot,\cdot),M(\cdot,\cdot)] &= 2\,(\ M(M(\cdot,\cdot)\,,\cdot) - M(\cdot,M(\cdot,\cdot))\ )\ .
\end{aligned} \tag{4.24}$$

The expressions here are, respectively,

1. application of a function to its argument
2. commutator of two operators
3. commutator (a measure for the failure of M to define a commutaive product)
4. 'derivator' (a measure for the failure of μ to be a derivation of M)
5. associator (a measure for the failure of M to define an associative product).

Now it is easy to understand the general case where the form degrees may be odd: The above commutators simply become graded commutators and the derivator becomes a graded derivator in the familiar way. Note that the associator remains intact if for M the product (4.17) is used, so that

$$[M,M] = 0 \tag{4.25}$$

(if the product on any internal degrees of freedom is associative).

With this in hand some intersting things about the exterior derivative on pull-back forms on loop space can be said:

If from now on the letter M is reserved for the special product derivation (4.17) and if d denotes the obvious unary derivation, we can write

$$\mathbf{d} \oint(\omega_1,\ldots,\omega_n) = \oint(d+M)(\omega_1,\ldots,\omega_n)\ . \tag{4.26}$$

The square is

$$\mathbf{d}^2 \oint(\omega_1,\ldots,\omega_n) = \oint\left(\frac{1}{2}\,[d,d] + \frac{1}{2}\,[M,M] + [d,M]\right)(\omega_1,\ldots,\omega_n)\ . \tag{4.27}$$

Using the above insight we find that all three terms vanish by themselves:

$$\begin{aligned}
\frac{1}{2}\,[d,d](\omega_1) &= dd\omega_1 = 0 \\
\frac{1}{2}\,[M,M](\omega_1,\omega_2,\omega_3) &= (-1)^{|\omega_1||\omega_2|}\left((-1)^{|\omega_1|}\omega_1\wedge\omega_2\right)\wedge\omega_3 - (-1)^{2|\omega_1|}\omega_1\wedge\left((-1)^{|\omega_2|}\omega_2\wedge\omega_3\right) = 0 \\
[d,M](\omega_1,\omega_2) &= (-1)^{|\omega_1|}\,d(\omega_1\wedge\omega_2) + (-1)^{|\omega_1|+1}\,(d\omega_1)\wedge\omega_2 + (-1)^{2|\omega_1|}\,\omega_1\wedge(d\omega_2) \\
&= (-1)^{|\omega_1|}\left(d(\omega_1\wedge\omega_2) - (d\omega_1)\wedge\omega_2 - (-1)^{\deg(\omega_1)}\,\omega_1\wedge(d\omega_2)\right) = 0\,,
\end{aligned} \tag{4.28}$$

due to the fact that the ordinary exterior derivative is nilpotent and a graded derivation of the ordinary wegde product, which is associative.

Further Hochschild operators. In [6] it was argued that one should consider generalizations of the multi-derivation $\phi_d + \phi_M$ using a unary 1-form A and a 0-ary 2-form B multi-derivation (thus using all possible derivations of grade 1), to obtain the derivation $\phi_d + \phi_M + \phi_A + \phi_B$. However, the motivation for this proposal used an argument which was a little shaky (for instance according to that argument there should have really been a term proportional to the field strength of A in equation (15) of [6]). But there is a way to derive this total derivation from an interesting loop space expression:

We know from the tensionful superstring [3] and in particular in the context of boundary state formalism [7] that the natural modification of the exterior derivative on loop space is a polar combination of the worldsheet supercharges, namely the object

$$\mathbf{d}_K \equiv \mathbf{d} + iT\,\iota_K\,, \tag{4.29}$$

where ι_K is the operator of inner multiplication with the loop space vector $K^{(\mu,\sigma)} = X'^{\mu}(\sigma)$, which gerenrates rigid reparameterizations, and we have kept the string tension T (a constant) for later discussion of the limit $T \to 0$.

Moreover [7, 8] it is known that the Wilson line of A along the string naturally generalizes to the multi-form

$$\mathcal{W}[X](\sigma,\sigma') \equiv \mathrm{P}\exp\left(\int_{\sigma}^{\sigma'} d\sigma\;\left(iA\mu\cdot X'^{\mu} + \frac{1}{2}\left(\frac{1}{T}F_A + B\right)_{\mu\nu} dX^{\mu}\wedge dX^{\nu}\wedge\right)(\sigma)\right)\mathbf{1} \tag{4.30}$$

where $\mathbf{1}$ denotes the constant unit 0-form on loop space.

This was the object of interest in [7]. However, in the present context it is worthwhile to generalize both this approach as well as the one in [6] and consider modified pull-back forms that have the above generalized Wilson line between each factor:

$$\oint_{(A,B)} (\omega)(\omega_1,\ldots,\omega_n) \equiv \int_{0<\sigma_i<\sigma_{i+1}<1\,\forall\, i} W(0,\sigma_1)\,\iota_K(\omega_1)(\sigma_1)\,W(\sigma_1,\sigma_2)\,\iota_K(\omega_1)(\sigma_1)\cdots W(\sigma_n,1)\;. \tag{4.31}$$

The physical interpretation of this object (if any) would of course remain to be understood (but that is necessarily true for all of the material presented here).

But the point of this definition is that the action of the modified exterior derivative $\mathbf{d}_K$ on this object in a certain scaling limit reproduces the action of the multi-derivations proposed in [6] up to an extra term:

If we let B scale as $1/T$ then

$$\mathbf{d}_K \oint_{(A,B)} (\omega_1,\ldots,\omega_n) = \oint_{(A,B)} (d+M+A+B)(\omega_1,\ldots,\omega_n) + \mathcal{O}(1/T)\;, \tag{4.32}$$

where the remaining terms of order $1/T$ have no further contractions with ι_K. Hence there is a scaling limit of *large* string tension $T \to \infty$ with $T\,B$ fixed in which Hofman's multi-derivation are obtained from a proper loop space differential.[2]

Applying $\mathbf{d}_K$ twice yields (*cf.* equation (23) in [6])

$$(d + M + A + B)(d + M + A + B) = \mathcal{H} + \mathcal{F} + \mathcal{N} + \mathcal{K} \tag{4.33}$$

where

$$\begin{aligned}
\mathcal{H} &= [d, B] + [A, B] \\
&= dB + A(B) \\
\mathcal{F}(\omega) &= \left([d, A] + \frac{1}{2}[A, A] + [M, B]\right)(\omega) \\
&= d(A(\omega)) + A(A(\omega)) + M(B, \omega) + (-1)^{|\omega|} M(\omega, B) \\
&= d(A(\omega)) + A(A(\omega)) - B \wedge \omega + \omega \wedge B \\
\mathcal{N}(\omega_1, \omega_2) &= ([d, M] + [A, M])(\omega_1, \omega_2) \\
&= (-1)^{|\omega_1|}\left(d_A(\omega_1 \wedge \omega_2) - (d_A\omega_1) \wedge \omega_2 - (-1)^{\deg(\omega_2)}\omega_1 \wedge (d_A\omega_2)\right) \\
\mathcal{K}(\omega_1, \omega_2, \omega_3) &= \frac{1}{2}[M, M](\omega_1, \omega_2, \omega_3) \\
&= (-1)^{|\omega_1||\omega_2|}\left((-1)^{|\omega_1|}\omega_1 \wedge \omega_2\right) \wedge \omega_3 - (-1)^{2|\omega_1|}\omega_1 \wedge \left((-1)^{|\omega_2|}\omega_2 \wedge \omega_3\right) .
\end{aligned} \tag{4.34}$$

$$[d + M + A + B, \alpha + \gamma] = d + M' + A' + B' \tag{4.35}$$

$$\begin{aligned}
\delta B &= [d + A, \alpha] + [B, \gamma] \\
&= d_A\alpha - \gamma(B) \\
\delta A(\omega) &= ([d + A, \gamma] + [M, \alpha])(\omega) \\
&= d_A(\gamma(\omega)) - \gamma(d_A(\omega)) + M(\alpha, \omega) + (-1)^{|\omega|} M(\omega, \alpha) \\
\delta M(\omega_1, \omega_2) &= [M, \gamma](\omega_1, \omega_2) \\
&= M(\gamma(\omega_1), \omega_2) + (-1)^{|\omega_1|} M(\omega_1, \gamma(\omega_2)) - \gamma(M(\omega_1, \omega_2))
\end{aligned} \tag{4.36}$$

[2] A $T \to \infty$ limit is somewhat unexpected since the appearance of the non-abelian 2-form is related to the appearance of tensionless strings arising as the boundaries of membranes that stretch between coinciding 5-branes. It remains to be seen if the above limiting procedure has any physical relevance or if it is just a formal curiosity. In principle nothing forbids to have membranes that stretch between some fixed point (another 5-brane or some singularity) and the stack of coinciding 5-branes far away. The resulting strings would indeed have large tension.

4.0.6 Polyakov's σ-model analogy

Polyakov observed [**?**] that the equations of motion of Yang-Mills theory make the dynamics of Wilson loops formulated on loop space have close similarity to ordinary σ-models with a group target:

The σ-model is defined by the action

$$S[g] = \mathrm{Tr}\int_{\mathcal{P}} (g^{-1}dg) \wedge \star(g^{-1}dg) \tag{4.37}$$

where

$$g : \mathcal{P} \to G \tag{4.38}$$

is a function from parameter space $\mathcal{P}$ to the group G. The classical equations of motion coming from this action are (*cf.* e.g. section 15 of [9])

$$d^\dagger(g^{-1}dg) = 0\,. \tag{4.39}$$

Since moreover $g^{-1}dg$ is a Lie(G)-vlaued 1-form the expression $g^{-1}dg$ can be regarded as a connection 1-form that is gauge equivalent to the trivial connection:

$$d(g^{-1}dg) + (g^{-1}dg) \wedge (g^{-1}dg) = 0\,. \tag{4.40}$$

When parameter space is identified with path space $\mathcal{P} = P_s^t(\mathcal{M})$ and the field g is identified with the holonomy along a path $g = W_A \in G$ this gives the two path space equations

$$\begin{aligned} &\mathbf{d}(W_A^{-1}\mathbf{d}W_A) + (W_A^{-1}\mathbf{d}W_A) \wedge (W_A^{-1}\mathbf{d}W_A) = 0 \\ &\mathbf{d}^\dagger(W_A^{-1}\mathbf{d}W_A) = 0\,. \end{aligned} \tag{4.41}$$

The first says that $(W_A^{-1}\mathbf{d}W_A)$ is the 1-form of a flat connection on path space. This turns out to be a special case of a more general class of so-called *r-flat* connections on path space which are discussed in §**??** (p.**??**).

The second equation has been considered in (4.12) and is fulfilled whenever the gauge field A which defines the holonomies W_A satisfies the Yang-Mills equations of motion.

Polyakov pointed out that this is an intriguing formal similarity between ordinary σ-models and Yang-Mills theory formulated in path/loop space which might even explain or at least be related to the integrability properties recently found in $N = 4$ SYM [**?**].

Related recent results as well as further references can be found in [**?**].

5. Physical applications

5.1 Flux Line Description of ordinary Gauge Theory

Consider ordinary gauge theory with gauge group G. Flux lines $q_1 \xrightarrow{\gamma} \bar{q}_2$ between two quarks (paths up to thin homotopy) are associated with group elements $W(\gamma)$. The question "How does $W(\gamma)$ transform as we move γ?" is answered by a 2-connection in a 2-bundle:

Let base 2-space be $\mathcal{P}(M)$, the space of paths up to thin homotopy in M naturally interpreted as a 2-space. Let the strict structure 2-group be given by $H = G$, $t = \mathrm{i}d$, $\alpha = \mathrm{A}d$ with G the above Lie group. Identify the 1-form A_i in the 2-connection with that from the above gauge theory.

The 2-holonomy in this 2-bundle associates group elements $\mathcal{W}(\Sigma)$ to the surface swept by γ and bounded by γ_1 and γ_2 such that $W(\gamma_2) = \mathcal{W}(\Sigma)\, W(\gamma_1)$.

The YM equations of motion of the original theory become the divergence freedom of the 2-connection 1-form. This is Polyakov's σ-model description of gauge theory §4.0.6 (p.29).

The 2-bundle context globalizes this approach and adds a new level of description. Gauge transformations of type 2 in the 2-bundle change the original bundle and its connection. Hence an isomorphism class of such 2-bundles describes a collection of gauge theories. In particular bundles with a "nonabelian twist" are included.

Once the isomorphism classes of these 2-bundles are understood one should see what they imply in terms of the ordinary gauge theories they come from.

This simple setup should be the right starting point for describing boundary states with 2-bundles.

For this and other reasons one will want to have operators acting on the space of 2-sections of a 2-bundle. This space is pretty much given by the 2-(pre)sheaf of 2-sections.

This space is a category with objects being 2-sections (functors) and morphisms being natural transformation between these. These natural transformations are essentially the gauge transformations in the original gauge theory. They are defined by maps from the point space of base space into H.

An operator on this should be a functor into another space of 2-sections of some 2-bundle over the same base 2-space. For instance it seems possible to map $W(\gamma)$ to $W(\gamma)^{-1}\, \mathbf{d}W(\gamma)$ in a functorial way by defining a suitable composition operation in the target.

However this seems not to be captured by 2-vectors.(?) Here is an attempt:

Let $(q, W(\gamma))$ and $(q', W(\gamma'))$ be the 2-group elements associated two composable paths γ by some 2-section. We are interested in mapping these in a functorial way to something involving the objects $W^{-1}(\gamma)\, \mathbf{d}W(\gamma)$.

So define a category with one object and morphisms given by pairs (g, y) with $g \in G$ and $y \in \mathfrak{h}$ with composition given by

$$(g, y) \circ (g', y') \equiv (gg', g'^{-1} y g' + y')\,.$$

Then

$$(q, W(\gamma)) \mapsto (W(\gamma)\,, W^{-1}(\gamma)\, \mathbf{d}W(\gamma))$$

is functorial since composition is respected

$$\begin{aligned}
&(q, W(\gamma)) \circ (q', W(\gamma')) = (q, W(\gamma \circ \gamma'))\\
\mapsto\ &(W(\gamma)\,, W^{-1}(\gamma)\, \mathbf{d}W(\gamma)) \circ (W(\gamma')\,, W^{-1}(\gamma')\, \mathbf{d}W(\gamma'))\\
&= (W(\gamma \circ \gamma')\,, W^{-1}(\gamma')\, W^{-1}(\gamma)\, (\mathbf{d}W(\gamma)) W(\gamma') + W^{-1}(\gamma')\, \mathbf{d}W(\gamma'))\\
&= (W(\gamma \circ \gamma')\,, W^{-1}(\gamma \circ \gamma')\, \mathbf{d}W(\gamma \circ \gamma'))
\end{aligned}$$

and identity morphisms are sent to the identity morphism:

$$(q, 1) \mapsto (1, 0)$$

Essentially, this is a 2-group with H being the abelian group of vector addition in the Lie algebra.

5.2 Strings in Kalb-Ramond backgrounds

A 2-bundle with trivial base 2-space and strict structure 2-group ($H = U(1), G = 1$) has an abelian gerbe of sections. This is known to describe the coupling of F-strings to the Kalb-Ramond 2-form. By §2 (p.3) the 2-holonomy in this 2-bundle computes the action of the string's worldsheet due to this background.

5.3 Dimensional reduction of Self-Dual 2-Form theory

An abelian 2-form theory in six dimensions with self-dual 3-form field strength gives rise to ordinary abelian YM (EM) in five dimensions upon compactification on a circle.

Using the facts from §3 (p.19) this has a pretty straightforwad generalization to a nonabelian 2-form theory with self-dual field strength in six dimensions. This does not seem to quite yield ordinary nonabelian YM, though.

6. 2-Sections

An ordinary bundle has a sheaf of sections. There should hence be a notion "2-section" and "2-sheaf" such that a 2-bundle has a 2-sheaf of 2-sections.

Apparently a gerbe is a special case of a 2-presheaf of a 2-bundle and with the appropriate definition it should in fact be a special case of a 2-sheaf of 2-sections of a 2-bundle.

Definition 6.1 *A* **2-section** σ *of a 2-bundle* $E \xrightarrow{p} B$ *is a 2-map* $B \xrightarrow{\sigma} E$ *such that* $B \xrightarrow{\sigma} E \xrightarrow{p} B$ *is naturally isomorphic to the identity 2-map on* B*.*

Before giving the definition of 2-presheaf recall that of an ordinary presheaf:

Definition 6.2 *The* **category of open subsets** *of a topological space* X*, denoted* $O(X)$*, has open subsets of* X *as objects and inclusions* $i : U \hookrightarrow V$ *as arrows.*

Definition 6.3 *A* **presheaf** *over* X *is a contravariant functor*

$$P : O(X)^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

from the category of open subsets of X *(def. 6.2) to* Set*.*

Hence a presheaf is a morphism in Cat. Its categorification should be a morphism in 2Cat, namely a 2-functor between 2-categories:

Definition 6.4 *Given a 2-space* S *the* **2-category of open subspaces** *of* S*, denoted* $O(S)$*, is the category whose*

- *objects are open subspaces* $U_i \xrightarrow{i_i} S$
- *morphisms are 2-maps* $U_i \to U_j$ *such that*

$$U_i \xrightarrow{i_i} S \Leftrightarrow U_i \longrightarrow U_j \xrightarrow{i_j} S$$

- *2-morphisms are natural transformations between these 1-morphisms.*

(Thanks to Toby Bartels for helpful comments on this definition).

Definition 6.5 *A* **2-presheaf** *is a contravariant 2-functor*

$$P : O(S)^{\mathrm{op}} \longrightarrow \mathrm{Cat}\,.$$

Every 2-bundle has a 2-presheaf (def. 6.5) of 2-sections (def. 6.1). Here the target category is that whose objects are 2-sections $U_i \to E$ and whose morphisms are natural transformations between these.

A. Lie 2-algebras

The following are some random notes on [10].

properties of 2-vector spaces :

$$\vec{f} \equiv f - i(s(f))$$

$$f = \left(x, \vec{f}\right)$$

$$\begin{aligned} s\Big((x, \vec{f})\Big) &= x \\ t\Big((x, \vec{f})\Big) &= x + t\Big(\vec{f}\Big) \end{aligned}$$

$$f \circ g = (s(f)\,, \vec{f} + \vec{g})$$

$$(x, \vec{f}) + (y, \vec{g}) = (x + y, \vec{f} + \vec{g})$$

$$\begin{aligned} \left[x \xrightarrow{f} y, a \xrightarrow{g} b\right] &= \Big([x, a]\,,\ \left[\vec{f}, a\right] + [y, \vec{g}]\Big) \\ &= \Big([x, a]\,,\ [x, \vec{g}] + \left[\vec{f}, b\right]\Big) \end{aligned}$$

Proposition A.1 *The 2-term L_∞ algebra asscoiated with the Lie 2-algebra of a strict Lie 2-group defined by the differential crossed module $(\mathfrak{g}, \mathfrak{h}, d\alpha, dt)$ is given by*

$$\begin{aligned} l_2(x_1, x_2) &= [x_1, x_2] \\ l_2(x, y) &= d\alpha(x)(y) \\ dy &= dt(y) \\ l_3(x_1, x_2, x_3) &= 0\,. \end{aligned}$$

.

Proof.
Let

$$\begin{aligned} g_i &= \exp(x_i) \\ h_i &= \exp(y_i)\ . \end{aligned}$$

Then the adjoint action is

$$\begin{aligned}
& (g_1, h_1) \cdot (g_2, h_2) \cdot (g_1, h_1)^{-1} \\
&= (g_1, h_1) \cdot (g_2, h_2) \cdot \left(g_1^{-1}, \alpha\left(g_1^{-1}\right)\left(h_1^{-1}\right)\right) \\
&= \left(g_1 g_2 g_1^{-1}, h_1 \alpha(g_1)(h_2)\, \alpha\left(g_1 g_2 g_1^{-1}\right)\left(h_1^{-1}\right)\right) \\
&\approx (\exp(x_2 + [x_1, x_2]\,, \exp(y_2 + [y_1, y_2] + d\alpha(x_1)(y_2) - d\alpha(x_1)(y_1)))) \\
&= (\exp(x_2 + [x_1, x_2])\,, \exp(y_2 + d\alpha(x_1 + dt(y_1))(y_2) - d\alpha(x_2)(y_1)))\ .
\end{aligned}$$

Hence

$$\begin{aligned}
[(x_1, y_1), (x_2, y_2)] &= ([x_1, x_2]\,, [y_1, y_2] + d\alpha(x_1)(y_2) - d\alpha(x_2)(y_1)) \\
&= ([x_1, x_2]\,, d\alpha(x_1 + dt(y_1))(y_2) - d\alpha(x_2)(y_1))\ .
\end{aligned}$$

Comparison with the proof of theorem 36 in [10] yields the given identifications

$$\begin{aligned}
l_2(x_1, x_2) &= [x_1, x_2] \\
l_2(x, y) &= d\alpha(x)(y) \\
dy &= dt(y) \\
l_3(x_1, x_2, x_3) &= 0\,.
\end{aligned}$$

□

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