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Higher Gauge Theory and its application to Strings and Elliptic Cohomology

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Abstract: The study of geometric realizations of elliptic cohomology has lead to intriguing relations between relativistic string physics and higher dimensional algebra. These naturally find their place in what is beginning to be called higher gauge theory, where the stringification of point particle dynamics is understood in terms of categorified algebraic structures. Here we propose merging the approach of a physically motivated category theorist with that of a mathematically motiviated string theorist in order to study the higher K-theory of categorified vector bundles and the parallel transport of nonabelian strings, in an effort to understand (enriched) elliptic objects in the context of higher quantum theory.

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1. Introduction

In modern physics a fruitful method for obtaining interesting higher order structures is the process of **stringification**, whereby theories of point particles are conceived as limits of theories of linearly extended strings that stretch between their endpoints.

In modern mathematics a fruitful method for obtaining interesting higher order structures is known as **categorification**, where algebraic structures are stringified by replacing objects by morphisms.

Recent developments taking place in the overlap of formal high energy physics and abstract higher dimensional algebra are indicative of a synthesis of these two. In particular, in what is beginning to be called **higher gauge theory** the dynamics of stringified point particles is studied using category theoretic insights.

The structures emerging thereby are beginning both to elucidate aspects of string theory as well as to become helpful tools in pure mathematics, as concerns notably the study of **geometric realizations of elliptic cohomology**.

Here we propose a project whose aim is to combine the approach of a physically motivated category theorist with that of mathematically motivated string theorist in order to develop a couple of existing partial constructions in elliptic cohomology into a coherent picture within higher gauge theory.

In particular, a central but still somewhat elusive concept in this context has turned out to be that of a **categorified vector bundle**. We shall begin by presenting and investigating several possible definitions of such structures and study their categorified K-theory, building on recent progress in the understanding of the categorification of *principal* fibre bundles and their relation to nonabelian gerbes, in which both applicants have been involved in their Ph.D. research. In particular, we propose a certain string-motivated refinement of one definition of categorified vector bundle which has already proven useful in the context of elliptic cohomology.

Categorified vector bundles ought to be just one puzzle piece in a grander scheme where supersymmetric quantum mechanics is systematically categorified to yield superstring physics. Judging from the relation of ordinary Dirac operators to K-theory it is reasonable to expect that what is needed for elliptic cohomology is a notion of categorified Dirac operator related to string supercharges, or in fact a categorification of the entire concept of a spectral triple.

Several aspects of such a point of view have already emerged in the literature, but its coherent development seems to be hampered by its intrinsic interdisciplinary nature, where crucial guidance is apparently to be found in string physics while crucial tools are provided only by category theory, two fields which are only beginning to be commonly recognized as closely related.

Recent developments in the context higher gauge theory have provided a bridge between these two fields which shall serve as the fundament for the project proposed here: the understanding of higher holonomy in categorified principal bundles ('2-bundles') has allowed to recognize the category theoretic structure underlying the well-known coupling of the string to the Kalb-Ramond field, and has for the first time provided a candidate formalism for describing nonabelian strings as they arise in configurations of M2-branes attached to M5-branes, thus possibly shedding light on the fundamental degrees of freedom in M-theory.

Urs Schreiber, working in string theory, has studied in his thesis the description of the superstring in terms of supersymmetric quantum mechanics on loop space, and, motivated by this, studied the theory of 2-holonomy in principal 2-bundles which describes the gauge theory of (non)abelian strings.

Igor Baković, working in category theory, has embedded in his thesis the 2-category of principal 2-bundles in the larger context of Grothendieck topologies for bicategories, 2-sheaves and 2-gerbes, thus illuminating crucial formal structures behind the concept of a 2-bundle.

Since elliptic cohomology can justly be expected to be in large parts about the classification of (vector) 2-bundles, describing global phenomena in string theory, we feel that our collaboration opens the interesting opportunity of merging expertise in string theory and in category theory to tackle this intriguing subject, which is possibly at the heart of M-theory as well as of much of modern geometry.

1.1 Background

There has for quite a while now been some cross-fertilization between string theory and category theory.

On the one hand the discovery of the description of open strings on D-branes in terms of derived categories of coherent sheaves and of Fukaya categories [1, 2] has shown that category theoretic language is inevitable for making progress with understanding the deeper structure of string theory's still somewhat elusive nature. Much of recent progress on mirror symmetry [3] and Seiberg-duality [4, 5, 6] would have been hardly conceivable without both the basic point of view provided by category theory, which for instance allows to identify a functor between D-brane states as such, as well as its sophisticated results, as for instance concerning Π-stability in triangulated derived categories [7, 8] of topological branes.

On the other hand, string theory has motivated a rich web of category-theoretic constructions in abstract mathematical physics and in pure mathematics. String theory's influence on the study of mirror symmetry is probably only the most commonly known of these. More archetypical and far-reaching is the description of topological, conformal or other field theories in terms of functors on cobordism categories. This originates in the simple picture of a quantum string (or p-brane) sweeping out its worldvolume, but is at the heart of fascinating results like the TFT construction of CFT correlators [9, **?**, **?**, **?**] and has lead to the conception of new structures like homotopy quantum field theories [**?**, **?**, **?**], which are being studied as rich mathematical objects in their own right.

But even in the face of this rather impressive list of currently known links between strings and categories, there are hints that a deeper, more systematic relation between stringification and categorification remains still to be revealed. Several of these hints have arisen in attempts of physicists and mathematicians to come to grips with geometric realizations if elliptic cohomology.

Elliptic cohomology is a generalized cohomology theory based on the concept of abstract *genus* [10]. Several striking but mystifying results about the properties of the **elliptic genus** were dramatically clarified when it was noted by Witten [11, 12] that the elliptic genus is essentially nothing but the index of the superstring's supercharge, which can be regarded as a Dirac operator on loop space. This lifts the relation between K-theory and the index of a supercharge in supersymmetric quantum mechanics from points to strings.

While Witten's result was regarded as a breakthrough, there were issues with making his argument rigorous and with using this insight for the study of elliptic cohomology proper. In a reaction to that situation Segal famously suggested in [10] the concept of an **elliptic object**. In generalization of how an ordinary vector bundle with connection can be regarded as a functor from paths to the category of vector spaces, an elliptic object should be an assigment of vecor spaces to loops and of linear operators to cobordisms between loops, just as in the functorial description of conformal field theory [13].

In this spirit, it was later proposed by Baas, Dundas and Rognes [14] that it should be possible to go from K-theory to elliptic cohomology by categorifying the concept of a vector bundle and studying equivalence classes of the resulting '2-vector bundles', thus essentially stringifying the study of ordinary fibre bundles, which describe charged point particles, to that of '2-bundles', which describe charged strings. This was checked to be essentially true, even though the definition of categorified vector bundle seems to admit further refinement, to be discussed in more detail below in §2.1 (p.8).

In what might be regarded as a synthesis of these two approaches, Stolz and Teichner have begun approaching this issue by refining Segal's concept of an elliptic object [15]. They note that supersymmetric quantum mechanics, regarded as a functor from (super-)'worldline' intervals to the category of (super-)vector spaces, encodes K-theory data, and then proceed to categorify this situation, obtaining a 2-functor on a 2-category of bounded surfaces. The resulting **enriched elliptic objects** map surfaces to 2-morphisms of a 2-vector space and can, in a currently controllable special case, be successfully related to elliptic cohomology.

While all three of these approaches currently fall short of completely capturing elliptic cohomology, they seem reasonably symptomatic of a grander underlying scheme, which points to the need for a more thorough understanding both of a systematic categorification of (supersymmetric) quantum mechanics of point particles, as well as of the concept of a vector bundle.

It should be noteworthy that these need not be two different tasks. One way to characterize an ordinary vector bundle is as a projectively generated module of the algebra of functions on its base space, a definition which is, incidentally, suitable for the generalization to noncommutative geometry. Such an algebra of functions, however, is already part of the data coming with a system of supersymmetric quantum mechanics with this space as its configuration space, and hence categorifying the latter is likely to yield a good categorification of the former.

Indeed, as has been particularly emphasized by Froehlich et al. [16, 17], the data defining supersymmetric quantum mechanics is precisely that describing spectral (not necessarily commutative) geometry [18]. Froehlich as well as Chamseddine [19, 20] have taken this point of view seriously and have discussed possible ways to lift it from point particles to strings, in a way that is quite in resonance with the philosophy of the approaches to elliptic cohomology mentioned above. Further investigations along the lines of these proposals for spectral stringy geometry seem to have been thwarted at their time by the immense attention that was being paid to the newly discovered ubiquity of the noncommutative aspect of noncummutative spectral geometry in string physics, as it appears in Matrix Theory and in the study open strings in Kalb-Ramond backgrounds.

But meanwhile, undisturbed by the winds of fashion in string theory, necessary ingredients for a categorification of quantum mechanics were being pondered by Baez et al. [21], who investigated the notion of a categorified Hilbert space. While motivated by purely formal considerations and the idea that concepts in physics eventually ought to be categorified [22, 23], such categorified vector spaces turn out to reappear in Baas, Dundas and Rognes' work on elliptic cohomology, as well as in the context of quiver gauge theories in string theory. This is further discussed in §2 (p.8).

The same school of thought has recently lead to the first systematic approach towards 2-bundles [24] and, building on that, to the study of the relation of 2-bundles to nonabelian gerbes and of connections and holonomy in such 2-bundles [25, 26].

In this context it was found in [27] that a certain strict 2-group called $\mathcal{P}_1\mathrm{Spin}(n)$ has all the right properties to be the structure 2-group for 2-bundles that capture the parallel transport of spinning strings. A direct relation of this result to the approach by Stolz and Teichner seems inevitable, but remains to be investigated. This is further discussed in §2.2 (p.16).

On the other hand, nonabelian 2-bundles with 2-connection are an interesting candidate for the description of M2-branes ending on M5-branes, which was rather recently speculated to be related to elliptic cohomology by Kriz and Sati in a series of papers [28, 29, 30, 31]. While the developments mentioned so far all pertain to the worldsheet description of strings, Kriz and Sati's work gives a new twist to the relation between string physics and elliptic cohomology by placing elliptic cohomology at the heart of the broader understanding of string theory which has emerged after the so-called second superstring revolution and which is famously known under the working title 'M-theory'.

Note: all text still very sketchy *at least* from here on.

When the process of stringification is applied twice, strings turn into membranes and their endpoints turn into endstrings. Like the endpoint of an open string may couple to a nonabelian bundle, the endstring of an open membrane ending on a collection of 5-branes is expected to couple to a nonabelian 2-bundle, or gerbe. Kriz and Sati argue in [28] that the worldsheet of such an 'endstring' should be closely related to the elliptic curves featuring in elliptic cohomology.

An argument for a special role played by elliptic cohomology in the context of full M-theory is given in [29]. There the authors note that due to common wisdom the RR-fields are objects in (twisted) K-theory, while the corresponding $NSNS$-fields are ordinary p-forms. But for IIB string theory, which is self-S-dual with these fields interchanging their roles, this is inconsistent. Consequently, they argue, both the RR fields as well as the NSNS fields must be regarded as objects in a higher coholomoly theory, namely in elliptic cohomology.

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1.2 Detailed Overview of the Project

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Concrete research tasks:

1. **Refinement of the definition of categorifed vector bundles, study of their categorified K-theory and its relation to elliptic cohomology** (§2.1 (p.8))

   (a) it is expected that elliptic cohomology is something like the higher K-theory of categorified vector bundles, but first steps along these lines [14] are very promising but also obviously not fully satisfactory

   (b) since categorification is a little like quantization in that, while being a systematic process, its result may depend on the choice of one of several otherwise equivalent starting points, the task is to see if their are 'better' categorifications of the notion vector bundle and to study their properties

2. **Study of 2-connections in $\mathcal{P}_1\mathrm{Spin}(n)$-2-bundles and examination of their relation to enriched elliptic objects** (§2.2 (p.16))

   - a principal 2-bundle $E \downarrow M$ with structure group $\mathcal{P}_1\mathrm{Spin}(n)$ was shown in [32] to yield an ordinary $\mathrm{String}(n)$-bundle $|E| \downarrow M$ after taking its nerve
   - such E hence gives a Lie (smooth) realization of $|E|$, which is known to describe spinning strings [33]
   - a proof that $\mathcal{P}_1\mathrm{Spin}(n)$-2-bundles on M are obstructed by $p_1/2$ (like $\mathrm{String}(n)$-bundles) was sketched in [26] but needs to be made precise
   - according to [25, 26] a 2-connection in a $\mathcal{P}_1\mathrm{Spin}(n)$-2-bundle is a 2-functor $\mathcal{P}_2(M) \to \mathcal{P}_1\mathrm{Spin}(n) - 2\mathrm{Tor}$. This looks tantalizingly similar to what Stolz and Teichner call an enriched elliptic object [15], but the relation needs to be worked out in detail

Further topics that should eventually be addressed:

1. **Parallel transport of endstrings of M2-branes on M5 branes and its possible relation to elliptic cohomology**

   [28]

2. **Examination of the possibility of categorifying the concept of a spectral triple**

   [34]

2. Technical Details

Here more details for what exactly we are to investigate in our project will be given. Right now what follows in §2.1 (p.8) is a discussion of the issue of vector 2-bundles as it appeared in US's thesis in section 4.4.

2.1 2-NCG and Derived Category Description of D-Branes

A *field* in physics is something that locally looks like a function on spacetime, but which really is a *section of a bundle*. For instance a spinor field looks like a spinor-valued function locally, but is really, globally, a section of a spinor bundle.

The quanta of ordinary fields are "point particles". Strings, on the other hand, are the quanta of what is called the **string field** (e.g. [35, 36, 37, 38, 39]).

Often, these string fields are treated as nothing but *functions* on space-time that take values in a vector space spanned by the excitation modes of the single first-quantized string. One would expect that, more precisely, globally these string fields should be generalized sections of some generalized notion of fibre bundle, instead.

At least to some extent this is captured by using ordinary fibre bundles on loop space, as we have reviewed in §**??** (p.**??**). Given our discussion in §**??** (p.**??**) and §**??** (p.**??**) on how, in the case of (uncharged) spinors on loop space, this is related to 2-bundles, it is tempting to guess that, more generally, a string field should be a 2-section of some 2-vector 2-bundle.

This is to some degree motivated by our discussion of the relation of the RNS string to supersymmetric quantum mechanics on loop space in §**??** (p.**??**). Given the close relation of SQM to noncommutative spectral geometry (NCG), we can consider states of a supersymmetric particle to be sections of a vector bundle, which arises as a finitely generated projective module of the algebra A of functions on configuration space. It seems to be a plausible conjecture that there is a categorification of this scenario which exhibits string fields as sections of some sort of "2-vector 2-bundle" that arises as a module for a "2-algebra of 2-functions". In fact, aspects of such a setup have been considered in the literature, as discussed below.

In part III we will exclusively deal with principal 2-bundles, since the generalization to the categorification of associated and vector bundles remains to be better understood. But, as a motivation for the general philosophy relating stringification with categorification that emerges from the considerations presented here, and as an outlook for further studies, we would like to sketch in the following some existing approaches and some further observations concerning vector 2-bundles, string fields, categorified supersymmetric quantum mechanics and noncommutative geometry and a possible relation to the description of D-brane states in terms of derived categories.

2.1.1 2-NCG

Ordinary Noncommtative Geometry (NCG) starts with the **Gelfand-Naimark theorem**, which says that a topological space is equivalently encoded in the C^*-algebra of continuous complex-valued functions over it. In the present context we wish to think of such a space as the configuration space of some particle. Upon "stringification" this particle is expected to become a linearly extended entity. Its configurations, when suitably interpreted, include the position of its endpoints together with a specification of how it stretches from one endpoint to the other. The collection of this data, a set of points (objects) and a set of strings (morphisms) between them, may form a category.

Therefore a natural question is whether there is a generalization of the Gelfand-Naimark theorem from sets to categories and if it can serve as a basis for a categorification of all of NCG – and how the result is related to string theory.

The answer to the first part of this question is positive, at least in the case where the underlying spaces are discrete. This, and the idea of categorified Hilbert spaces (which would be the second ingredient in a categorified spectral triple) was discussed in [21].

And indeed, it seems that starting from such a categorified GN theorem and following the logic of categorified NCG one does arrive at descriptions of string physics, as discussed further in §2.1.2 (p.11).

The starting point for turning geometry into algebra is that spaces may be characterized by algebras of functions over them. For instance, topological spaces are characterized by C^*-algebras of continuous functions (the Gelfand-Naimark theorem) and measure spaces by von Neumann algebras of bounded measurable functions.

In each case points of the space X are recovered in terms homomorphisms from the algebra of functions $K^X \equiv \{f : X \to K\}$ to K itself: For every $x \in X$ we get a homomorphism $\tilde{x} \colon K^X \to K$ by setting

$$\begin{aligned} \tilde{x} : K^X &\to K \\ f &\mapsto f(x) \; . \end{aligned}$$

When categorifying, spaces becomes 2-spaces (categories whose point and morphism spaces are topological spaces, or measure space, etc.) and functions become functors.

Let Q be any 2-space and let $\mathcal{K}$ be any monoidal category. The functor category $\mathcal{K}^Q$ (*cf.* §**??** (p.**??**)) now indeed encodes not just the point space of Q, but also the arrow space:

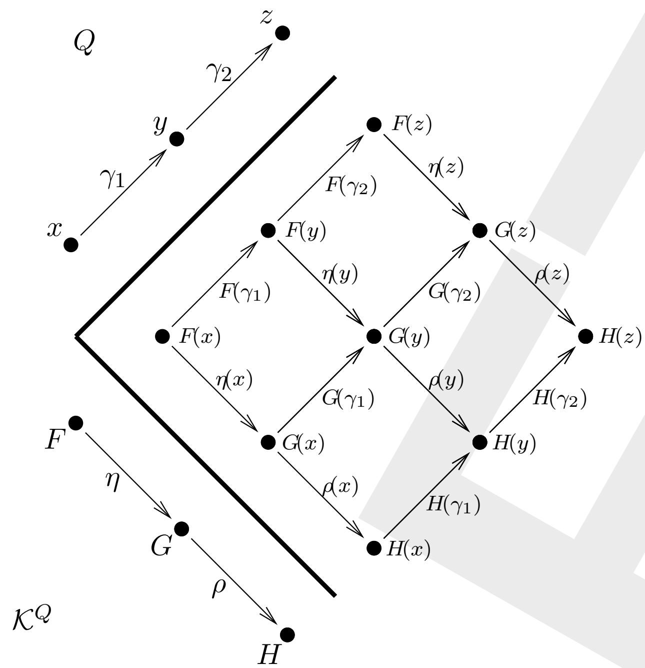
Every point $x \in \mathrm{Ob}(Q)$ gives rise to a functor $\tilde{x}$ defined by

$$\begin{aligned} \tilde{x} : \mathcal{K}^Q &\to \mathcal{K} \\ (F \xrightarrow{\eta} G) &\mapsto \left(F(x) \xrightarrow{\eta(x)} G(x) \right) \end{aligned}$$

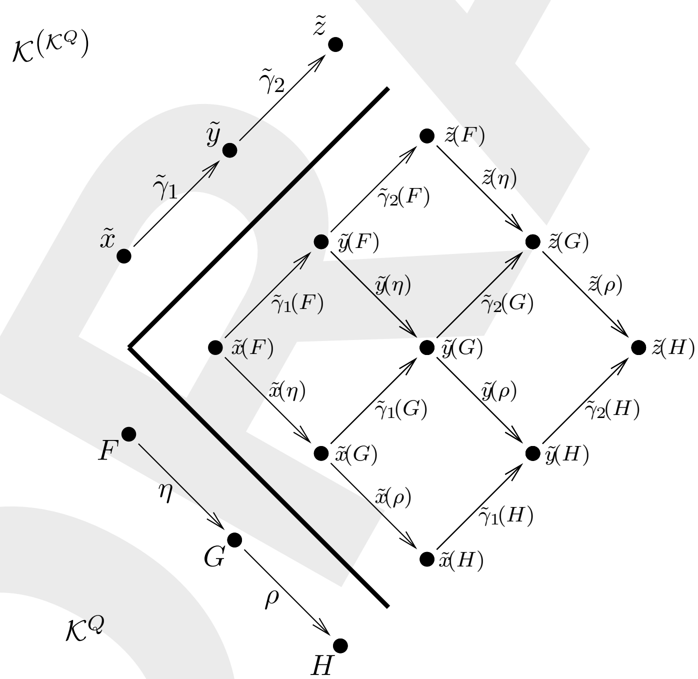
and every arrow $\gamma : x \to y$ in $\mathrm{Mor}(Q)$ gives rise to a natural transformation $\tilde{\gamma}$ between these functors

$$\begin{aligned} \tilde{\gamma} : \tilde{x} &\to \tilde{y} \\ \tilde{x}(F) &\xrightarrow{F(\gamma)} \tilde{y}(F) \; . \end{aligned}$$

This is best seen by looking at some naturality squares. Here is a 2-space Q together with a chain of three functors $F \to G \to H$ from Q to $\mathcal{K}$:



By using the definition of $\tilde{x}$ and $\tilde{\gamma}$ from above this can be relabelled equivalently to look like a 2-space with three points F, G and H and a chain $\tilde{x} \to \tilde{y} \to \tilde{z}$ of three functors from this to $\mathcal{K}$:



This duality is at the heart of the categorified Gelfand-Naimark theorem [21].

2.1.2 Vector 2-Bundles

Physical fields are in general not just functions on parameter space (spacetime/worldvolume) but are **sections of fibre bundles** over parameter space. Similarly, the wave function itself is in general not a function on configuration space, but a section of some bundle over configuration space. A central theme in categorified SQM is therefore necessarily that of 2-bundles.

It is well known (and reviewed in [1, 2]) that states of open strings ending on D-branes are described by certain **derived categories** (of coherent sheaves or of quiver representations). In §2.1.2.4 (p.14) it is discussed how this might naturally be interpretable in terms of categorified wave functions taking values in a **line 2-bundle**.

There are several equivalent descriptions of ordinary vector bundles. It turns out that the categorification depends on which description one starts with.

When using the definition which says that the typical fibre of a vector bundle is a vector space one ends up with categorifying the concept of a vector space itself. This was done in [32]. The **2-vector spaces** obtained this way are the right concept for instance for discussing Lie 2-algebras (§**??** (p.**??**)) but they do not seem to give rise to an interesting notion of vector 2-bundle.

The definition of vector bundles most natural in the NCG context is that saying that a vector bundle $E \to M$ is a finitely generated projective module of the C^*-algebra $\mathbb{C}^M$ of complex-valued continuous functions on M. Categorifying this definition amounts to categorifying $\mathbb{C}^M$ such that the result is what should be called a 2-ring.

2.1.2.1 Vector 2-Bundles as 2-Modules. Aspects of this problem have notably been addressed in [21]. There it was argued that a good categorification of $\mathbb{C}^M$ is a functor category Hilb^Q, where Q is some base 2-space replacing M and Hilb is the category of Hilbert spaces, replacing $\mathbb{C}$.

A slightly different but very similar idea is used in [40], where instead of the category Hilb the category Vect is used. This amounts simply to forgetting about the scalar product.

The crucial point is that the tensor product in Vect (Hilb) makes Vect^Q (Hilb^Q) into a monoidal category and indeed, at least in the case studied in [21], into something that deserves to be called a **2-algebra**.

In the spirit of this concept of categorified function algebras the authors of [14] defined a vector 2-bundle to be something that locally is similar to a bundle whose typical fibre looks like Vect^n, for some integer n.

Equivalence classes of ordinary vector bundles are described by K-theory. Therefore one would expect that equivalence classes of vector 2-bundles are described by some categorification of K-theory, which perhaps should be related to the elliptic genus.

In [14] however it was found that equivalence classes of the vector 2-bundles as defined there are not quite described by elliptic cohomology, even though by something the authors call a *form* of elliptic cohomology.

On the other hand, this is maybe not too surprising. One should note that Vect^Q is more like a categorification of functions taking values in the natural numbers, than in the complex numbers (compare the discussion in [22]). In particular, there are no additive or multiplicative inverses in Vect^Q. Due to that the "transition functions" in [14] are in general not invertible, for instance. This should mean that there must be a categorification of the notion "vector bundle" which more faithfully captures the crucial properties of ordinary vector bundles.

2.1.2.2 How to categorify function algebras? The above disucssion suggests that some more thoughts on the "right" categorification of function algebras is in order before vector 2-bundles can be addressed. One possibility to improve on Vect^Q might be the following:

We had observed that Vect^Q is lacking additive and multiplicative inverses. Hence we could try to enlarge Vect^Q by including such inverses, in a way similar to how one gets from the natural numbers to the integers and then the rational numbers.

In order to discuss this it turns out to be helpful to circumvent a couple of problems for the moment by restricting attention to base 2-spaces Q whose sets of points and arrows are *finite*. In particular, restrict attention to categories $Q = C_{(V,E)}$ which are free categories over finite directed graphs (V, E) (*cf.* §**??** (p.**??**)). This serves as the categorification of the concept of a space consisting of a finite number of points.

In this case it is a simple fact that the functor category Vect^Q is the same as the category $KQ-\mathrm{Mod}$ of (left, say) modules of the **path algebra** KQ of Q,

$$\mathrm{Vect}^Q = KQ-\mathrm{Mod}\,.$$

Here the path algebra KQ is the algebra freely generated by the set of morphisms in Q with the product between these generators defined to be their composition when composable and zero otherwise.

This equivalent reformulation suggests to use the tensor product over KQ in order to form a monoidal category. By this reasoning we are led to include multiplication and multiplicative inverses by going from $KQ-\mathrm{Mod}$ to $KQ-\mathrm{Mod}-KQ$, the category of KQ *bimodules* (over K). The multiplicative inverses in $KQ-\mathrm{Mod}-KQ$ give rise to a group known as the *Picard group* of KQ.

(In a more general context one might of course want to consider different algebras A, B and the category $A-\mathrm{Mod}-B$. By left monoidal multiplication the weakly invertible elements $T \in A\text{–Mod–}B$ give rise (if they exist) to a 'tilting equivalence' between $A\text{–Mod}$ and $\mathrm{Mod}-B$, in which case A and B are Morita equivalent.)

In a next step this category should be enlarged to allow a notion of subtraction. This again implies that given any object b there should be a way to 'decompose' it into abjects a and c. A diagramatic way to do this is by means of an exact sequence $a \to b \to c$. A slightly more general concept than this is that of a *distinguished triangle* in a triangulated category.

Using a triangulated category together with a **stability condition** on it [7, 8] subtraction is implemented by taking direct sums and then projecting the result onto the subset of objects which are **stable** with respect to this stability condition.

Now, a triangulated category is naturally obtained from $KQ-\mathrm{Mod}$ by passing to its **derived category** $\mathbf{D}(KQ-\mathrm{Mod})$. The derived category $\mathbf{D}(C)$ of any additive category C is like the category $\mathbf{Ch}(C)$ of chain complexes in C but modulo some identifications.

Hence we should choose a stability condition on $\mathbf{D}(KQ-\mathrm{Mod})$ and also pass from $KQ-\mathrm{Mod}-KQ$ to its derived category. $\mathbf{D}(KQ-\mathrm{Mod}-KQ)$.

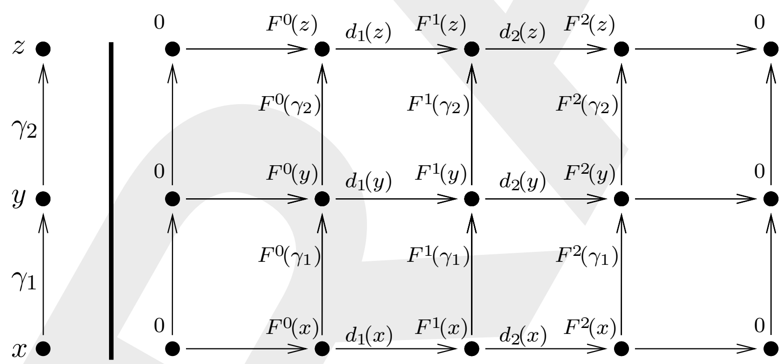
(The weakly invertible objects in $\mathbf{D}(KQ-\mathrm{Mod}-KQ)$ are known as *two-sided tilting complexes* and their isomorphism classes form a group known as the *derived Picard group* of KQ [41, 42, 43, 44].)

2.1.2.3 Vector 2-Bundles as $\mathbf{D}(A-\mathrm{Mod}-A)$-modules. This way we have arrived at the proposal that a 'good' categorification of an ordinary function algebra on the set $\mathrm{Ob}(Q)$ with values in K would be to replace $\mathrm{Ob}(Q)$ by Q and the function algebra by the monoidal category $\mathbf{D}(KQ-\mathrm{Mod}-KQ)$. Modules of this would locally look like $(\mathbf{D}(KQ-\mathrm{Mod}))^n$ for some integer n. These would be our proposed vector 2-bundles over Q.

One might be worried that by going to derived categories the simple idea that a categorified function on $\mathrm{Ob}(Q)$ is a functor on Q is lost. However, this is not the case. Namely, a chain complex of functors $Q \to \mathrm{Vect}$ is the same as a functor $Q \to \mathbf{Ch}(\mathrm{Vect})$,

$$\mathbf{Ch}\big(\mathrm{Vect}^Q\big) = (\mathbf{Ch}(\mathrm{Vect}))^Q$$

as the following diagram illustrates.

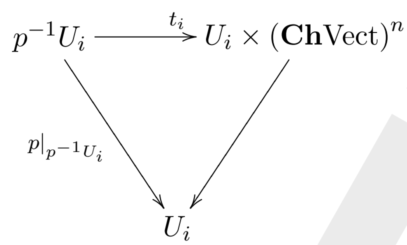


Here $x \xrightarrow{\gamma_1} y \xrightarrow{\gamma_2} z$ is a morphism in Q. The F on the right can be either regarded as giving a functor $Q \to \mathbf{Ch}\mathrm{Vect}$ or a chain complex

$$0 \to F^1 \xrightarrow{d_1} F^1 \xrightarrow{d_2} F^2 \to 0$$

of functors $F^i : Q \to \mathrm{Vect}$. That this diagram commutes can be regarded as a consequence of the definition of morphisms between functors (natural transformations) or as a consequence of the definition of chain maps. Moreover, due to the rule for 'vertical' composition of natural transformations (which is really going horizontal in this figure), we have $d_1(x) \circ d_2(x) = 0$ and similarly for y and z irrespective whether we regard this as a chain complex of functors or as a single functor into $\mathbf{Ch}(\mathrm{Vect})$.

This means that a **vector 2-bundle** E according to the above proposal would be a 2-bundle with typical fibre $(\mathbf{Ch}\mathrm{Vect})^n$, i.e. one with local 2-trivializations



(*cf.* §**??** (p.**??**)).

A 2-section of this bundle resticted to $U_i \simeq Q_i$ would be a functor from Q_i to $(\mathbf{Ch}\mathrm{Vect})^n$, i.e. an element in

$$\mathrm{Ob}\Big(\big((\mathbf{Ch}\mathrm{Vect})^{Q_i}\big)^n\Big) = \mathrm{Ob}((\mathbf{D}(KQ_i{-}\mathrm{Mod}))^n)\ .$$

On double overlaps $U_{ij} = Q_i \cap Q_j$ the 2-transition

$$\bar{t}_i \circ t_j|_{U_{ij}} : Q_{ij} \times (\mathbf{Ch}\mathrm{Vect})^n \to Q_{ij} \times (\mathbf{Ch}\mathrm{Vect})^n$$

must be an invertible $n \times n$ matrix of two-sided tilting complexes in $\mathbf{D}(KQ{-}\mathrm{Mod}{-}KQ)$.

More precisely, such a matrix should act componentwise by the usual formula for matrix multiplication using the derived product operation of $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod}{-}KQ_{ij})$ on $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod})$ and the direct sum operation in $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod})$ followed by a stability projection using the stability condition that we have chosen on $\mathbf{D}(KQ_{ij}{-}\mathrm{Mod})$ (in §2.1.2.2 (p.12)). These matrices would supersede the non-invertible transition matrices in [14].

The tractable special case of a **line 2-bundle** (i.e. a vector 2-bundle with $n = 1$) is already quite interesting:

2.1.2.4 Derived Category Description of D-Branes. If the general relation between categorification and stringification mentioned at the beginning of §**??** (p.**??**), as well as the notion of vector 2-bundle in §2.1.2.3 (p.13) are any good, then a 2-section of a line 2-bundle as described above should describe states of string, somehow.

And this indeed turns out to be the case.

By the above reasoning a 2-section of a line 2-bundle is locally an object of $\mathbf{D}(KQ{-}\mathrm{Mod}) = \mathbf{D}\big(\mathrm{Vect}^Q\big)$. This is indeed known to describe states of string stretched between D-branes, as reviewed in [1, 2].

In this context the path category Q, or rather its underlying graph, is really a quiver diagram encoding the precise nature of the moduli space of the effective field theory on these D-branes, or equivalently the nature of the transverse 'compact' dimensions. Hence it might seem that the analogy argued for above breaks down in that Q is not in any sense a categorified spacetime. But it turns out that viewed from a suitable perspective Q does have to be identified with a latticized spacetime after all, an effect known as **dimensional**

deconstruction. In particular, when Q is like $\mathbb{Z}_\infty$ or $\mathbb{Z}_\infty \times \mathbb{Z}_\infty$ it describes [45] two compactified dimensions of theories on 5-branes where 2-bundles are expected to play a role.

As discussed in §2.1.2.3 (p.13), gauge transformations of such a 2-section of a line 2-bundle have to be elements of the weak Picard 2-group inside $\mathbf{D}(KQ-\mathrm{Mod}-KQ)$. And indeed, these elements are known to describe duality transformations on these configurations, known as (fractional) Seiberg duality.

The relation of Seiberg duality to tilting equivalences is discussed in section 5.4 of [5]. Its relation to monodromies in moduli space, which goes back to [6] and others, is briefly reviewed in the introduction of [46].

2.2 Parallel Transport of Superstrings

(investigation of the relation between enriched elliptic objects and 2-connections in $\mathcal{P}_1\mathrm{Spin}(n)$-2-bundles:

both are 2-functors on $\mathcal{P}_2(M)$, both exist iff the first Pontryagin class vanishes, both describe parallel transport of spin aspect of strings)

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