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p-Functors from p-Paths to p-Torsors

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Abstract: Principal G_2-2-bundles with 2-connection and 2-holonomy are given locally by 2-functors $\mathrm{hol}_i \colon \mathcal{P}_2(U_i) \to G_2$, which are related on $(n+1)$-fold overlaps by 2-functor n-isomorphisms. Here we sketch the proof that a gauge equivalence class of such a collection of local 2-holonomy 2-functors is equivalently encoded in a single global 2-functor $\mathrm{hol} \colon \mathcal{P}_2(M) \to G_2{-}2\mathrm{Tor}$ from 2-paths in the base manifold M to the 2-category of G_2-2-torsors.
This is taken from section 12.4 of [1].

Contents

Warning: As usual, I do not indicate all re-whiskering that is necessary in order for the displayed diagrams to be well-formed. But at one point in this discussion it is crucial to make this re-whiskering explicit. This is discussed in the paragraph "The various 2-morphisms involved".

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1.1 p-Functors from p-Paths to p-Torsors

So far we have constructed p-holonomy making use of good coverings of the base manifold. This has the advantage that this way p-holonomy can locally be defined as nothing but a p-functor to the structure p-group. But the resulting global p-holonomy p-functor defined by gluing these local p-functors by p-functor n-morphisms on $(n+1)$-fold overlaps is not *manifestly* insensitive to the choice of good covering.

The reason for this is that a given fiber of a principal G_p-p-bundle is, while isomorphic to G_p, not *canonically* isomorphic to G_p. It is a *G_p-p-torsor* rather than G_p itself.

A nice pedagogical introduction to torsors can be found in [2]. The precise definition of (1-)torsors and 2-torsors is stated for instance in Toby Bartels' paper on 2-bundles [3].

Hence a holonomy functor should really associate to any path in the base manifold a *torsor morphism* between the fibers above the endpoints of the path. More precisely, a p-holonomy p-functor should be a p-functor from p-paths in the base manifold to p-morphisms of p-torsors.

$$\mathrm{hol}\colon \mathcal{P}_p(M) \to G_p{-}p\mathrm{Tor}\,.$$

Even more precisely, since we would want hol to be a smooth p-functor between smooth p-categories, while $G_p - p\mathrm{Tor}$ does not have a smooth structure, we should have a p-functor

$$\mathrm{hol}\colon \mathcal{P}_p(M) \to \mathrm{Trans}_p(E)\ .$$

Here $\mathrm{Trans}_p(E)$ is the p-category of **p-transporters** in a principal G_p-p-bundle $E \to M$ over a categorically trivial base manifold M. Objects of $\mathrm{Trans}_p(E)$ are the fibers E_x of E for all $x \in M$, regarded as G_p-p-torsors, and n-morphisms in $\mathrm{Trans}_p(E)$ are their p-torsor n-morphisms.

Given any such p-functor to $\mathrm{Trans}_p(M)$ we can always forget about the smooth structure and regard it as a p-functor to $G_p{-}p\mathrm{Tor}$. For notational convenience, this is what we shall do in the following.

Indeed, this more intrinsic description of global p-holonomy is *equivalent* to the one we have been concentrating on up to this point. In §1.1.1 (p.2) we recall the well-known way how this equivalence works for 1-bundles. Then in §1.1.2 (p.8) the generalization to 2-bundles is spelled out.

1.1.1 1-Torsors and 1-Bundles with Connection and Holonomy

A principal G-bundle with connection over a base manifold M is specified by a functor

$$\mathrm{hol}\colon \mathcal{P}_1(M) \to G\text{–}\mathrm{Tor}\,.$$

We shall now choose any good covering $\mathcal{U} \to M$ of M, $\mathcal{U} = \bigsqcup_{i\in I} U_i$, and demonstrate how this single functor encodes local holonomy functors

$$\mathrm{hol}_i\colon \mathcal{P}_1(U_i) \to G$$

on each U_i, which are related on double overlaps by natural transformations $\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$, as described in §**??** (p.**??**). In the process of doing so, the procedure for computing global

1-holonomy in terms of the hol_i, as described at the end of §**??** (p.**??**), drops out automatically.

For 1-bundles this is all rather simple and very well known. We find it worthwhile to restate these facts in order to make the generalization to 2-bundles in §1.1.2 (p.8) more accessible.

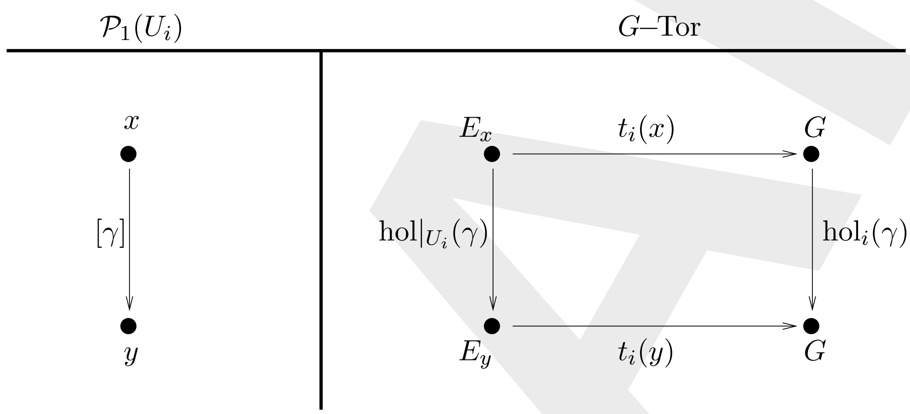
So let $\mathcal{U} \to M$ be a good covering of M, $\mathcal{U} = \bigsqcup_{i \in I} U_i$. When restricted to any of the U_i, the functor hol is naturally isomorphic to a local holonomy functor

$$\mathrm{hol}_i \colon \mathcal{P}_2(U_i) \to G\,.$$

Fix any such natural isomorphism

$$\mathrm{hol}|_{U_i} \xrightarrow{t_i} \mathrm{hol}_i\,.$$

It is specified by a naturality square

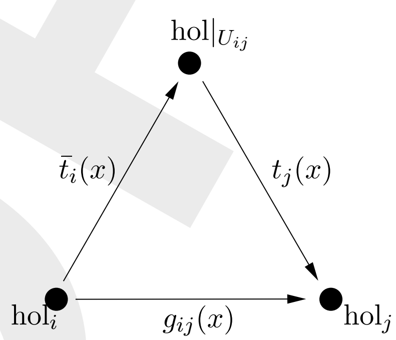


Here we use the fact that G itself is a G-torsor and that its automorphisms in G–Tor correspond to (right)-multiplication with elements in G.

It follows that on double overlaps we have natural isomorphisms

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$$

given by this commuting diagram



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Here $\bar{t}_i$ denotes the inverse of t_i. We write $\bar{t}_i$ because later on, when we categorify, t_i will be only weakly invertible (up to equivalence) and $\bar{t}_i$ will denote any of its weak inverses.

1.1.1.1 Line Holonomy in Terms of local Trivializations. Now consider the application of hol to any morphism $[\gamma] \in \mathcal{P}_1(M)$ that does not necessarily sit in a single patch U_i

$$\mathrm{hol}\left(x \xrightarrow{[\gamma]} y \right) = E_x \xrightarrow{\mathrm{hol}(\gamma)} E_y \ .$$

We can always split up $[\gamma]$ into pieces

$$x \xrightarrow{[\gamma]} y \ = \ x \xrightarrow{[\gamma_{i_1}]} y_{i_1} \xrightarrow{[\gamma_{i_2}]} y_{i_2} \cdots \xrightarrow{[\gamma_{i_n}]} y$$

such that $\gamma_{i_m} \in \mathcal{P}_2(U_{i_m})$, $\forall\, m$. Applying hol to that similarly yields

$$\mathrm{hol}\left(x \xrightarrow{[\gamma]} y \right) = E_x \xrightarrow{\mathrm{hol}(\gamma_{i_1})} P_{y_{i_1}} \xrightarrow{\mathrm{hol}(\gamma_{i_2})} P_{y_{i_2}} \cdots \xrightarrow{\mathrm{hol}(\gamma_{i_n})} E_y \ .$$

But at this point we can apply the commutativity of the above naturality square and set

$$\mathrm{hol}(\gamma_i) = t_i(x) \circ \mathrm{hol}_i(\gamma_i) \circ t_i^{-1}(y) \ .$$

This procedure is indicated in figure 1.

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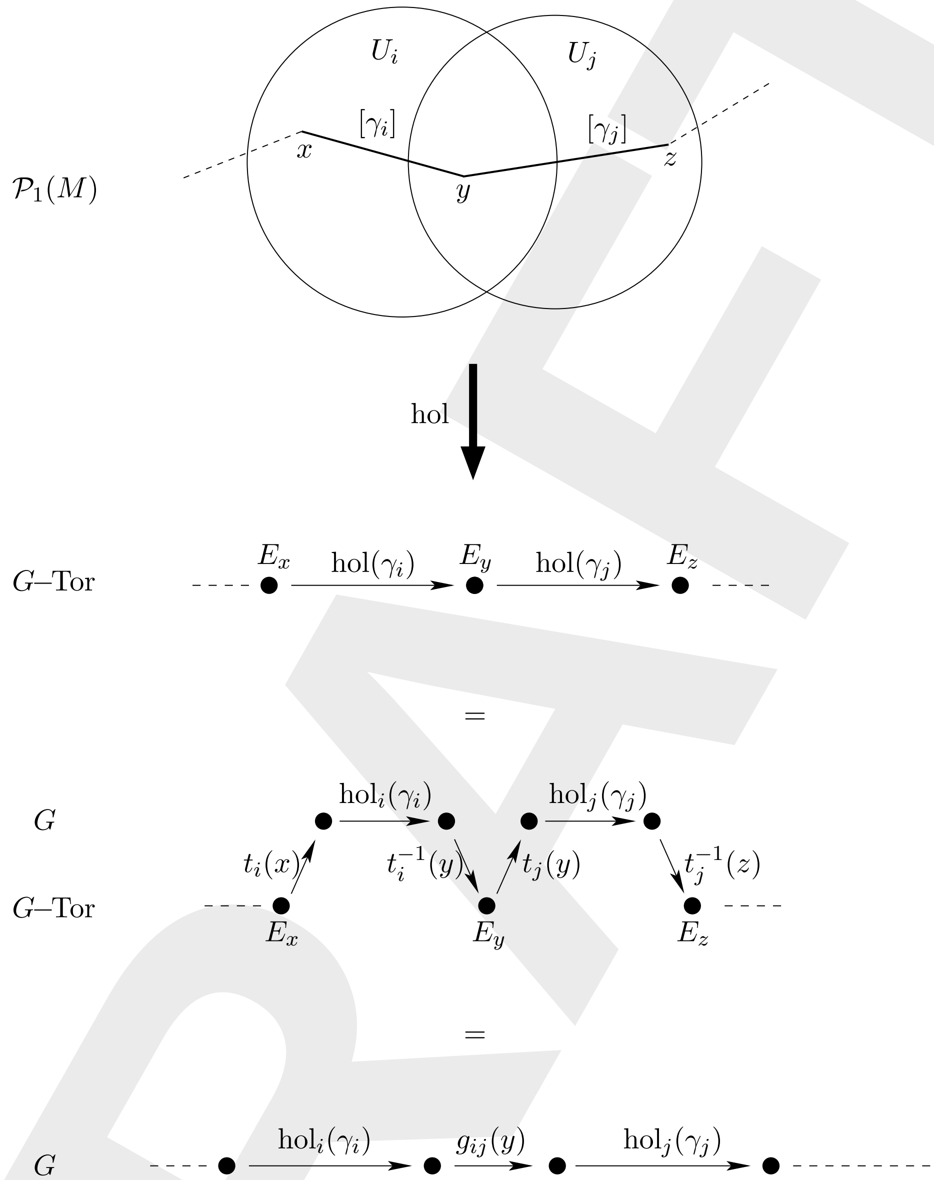


Figure 1: Global (1-)holonomy in terms of (1-)torsor (1-)morphisms. The functor hol associates torsor morphisms between fibers to paths in the base manifold. Using trivializations t_i on patches U_i these torsor morphisms can be identified with elements of the structure group. The step $t_i^{-1} \circ t_j$ from one trivialization to another one on double overlaps U_{ij} gives rise to multiplication by the transition function g_{ij}.

1.1.1.2 Gauge Transformations. Changing from one local trivialization

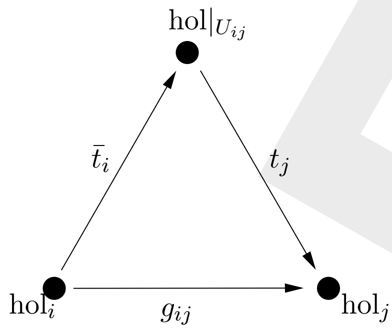
$$i \mapsto \left(\mathrm{hol}|_{U_i} \xrightarrow{t_i} \mathrm{hol}_i \right)$$

to another one

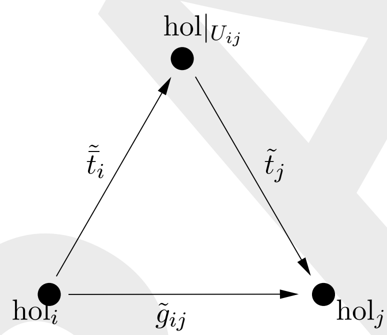
$$i \mapsto \left(\mathrm{hol}|_{U_i} \xrightarrow{\tilde{t}_i} \tilde{\mathrm{hol}}_i \right)$$

is called a **gauge transformation**.

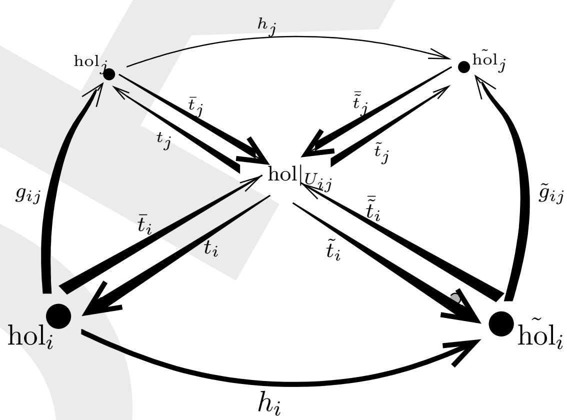
This corresponds to replacing the transition



by another transition $\tilde{g}_{ij}$ which is given by the commutativity of this diagram:



This situation is depicted by the following commuting diagram:



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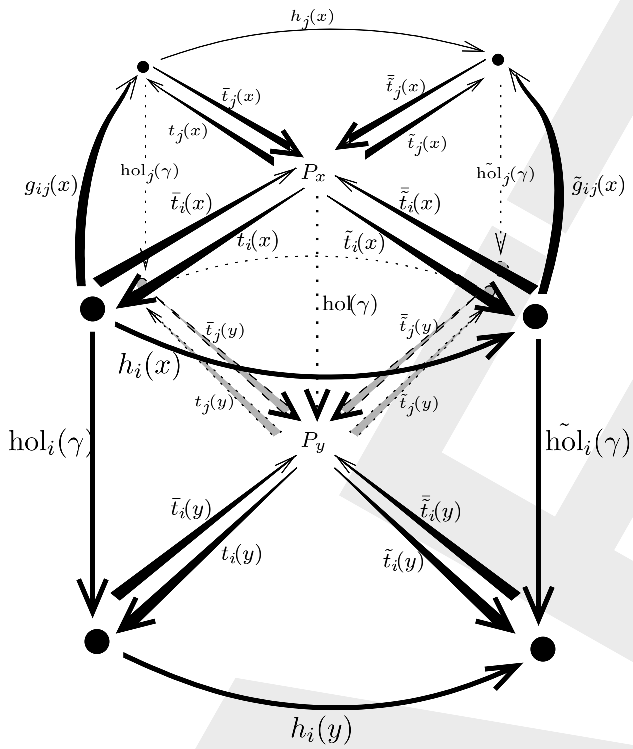
In components this means that the following diagram commutes, too:



From this diagram one reads off that

$$\tilde{g}_{ij} = h_i^{-1}\, g_{ij}\, h_j$$

and that

$$\tilde{\mathrm{hol}}_i = h_i^{-1}\, \mathrm{hol}_i\, h_i\,,$$

where

$$h_i = \bar{t}_i \circ \tilde{t}_i$$

and

$$h_j = \bar{t}_j \circ \tilde{t}_j\,.$$

This is the well-known gauge transformation law for 1-bundles with connection.

In fact, when read from the left to the right one sees that this diagram is equivalent to the diagram (**??**) (p. **??**) which depicts a natural isomorphism of the global holonomy 1-functor regarded as a functor

$$\mathrm{hol}\colon \mathcal{P}_1^C(\mathcal{U}) \to G$$

from the Čech-extended 1-path 1-groupoid (def. **??**, p. **??**) of the good covering $\mathcal{U}$ to the structure group G. This justifies our use of the same symbol "hol" for both functors.

In the next subsection all this is categorified.

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1.1.2 2-Torsors and 2-Bundles with 2-Holonomy

Let G_2 be any 2-group and let $G_2\!-\!2\mathrm{Tor}$ be the 2-category of G_2-2-torsors.

A principal G_2-2-bundle with connection and holonomy is specified by a 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to G_2\!-\!2\mathrm{Tor}\,.$$

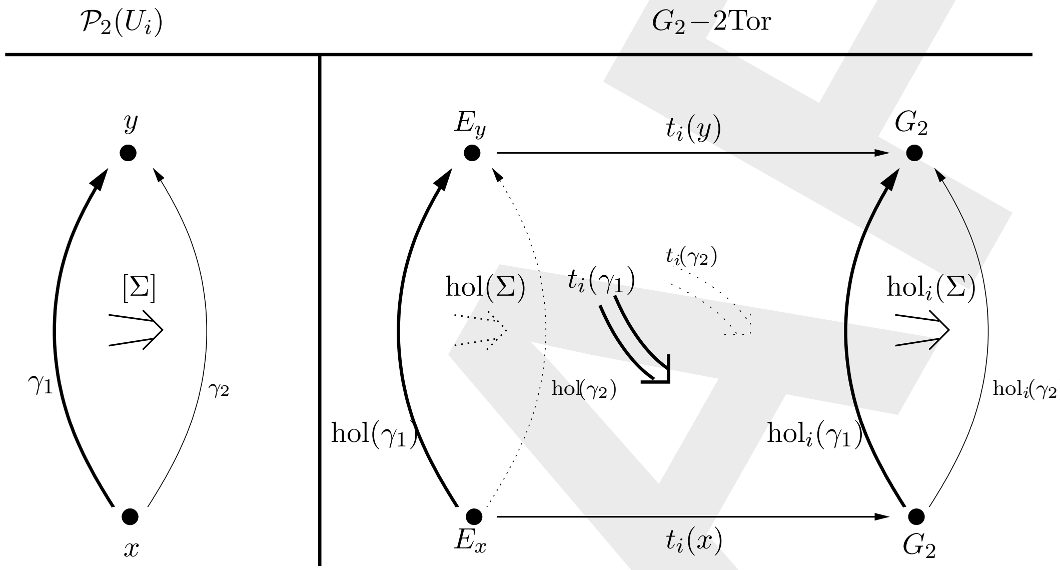
When restricted to any of the U_i, the 2-functor hol is pseudonaturally isomorphic to a local 2-holonomy 2-functor

$$\mathrm{hol}_i\colon \mathcal{P}_2(U_i) \to G_2\,.$$

Fix any such natural isomorphism

$$\mathrm{hol}|_{U_i} \xrightarrow{t_i} \mathrm{hol}_i\,.$$

It is specified by a naturality tincan diagram



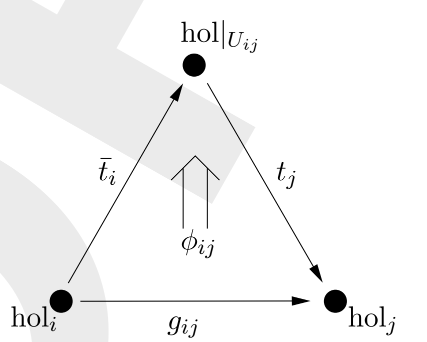
(1.1)

Here we use the fact that G_2 itself is a G_2-torsor and that its automorphisms in $G_2\!-\!2\mathrm{Tor}$ correspond to (right)-multiplication with identity morphism in G_2 and that its 2-morphisms correspond to 2-torsor 2-morphisms between these.

It follows that on double overlaps we have pseudonatural isomorphisms

$$\mathrm{hol}_i \xrightarrow{g_{ij}} \mathrm{hol}_j$$

given by diagrams like this:



(1.2)

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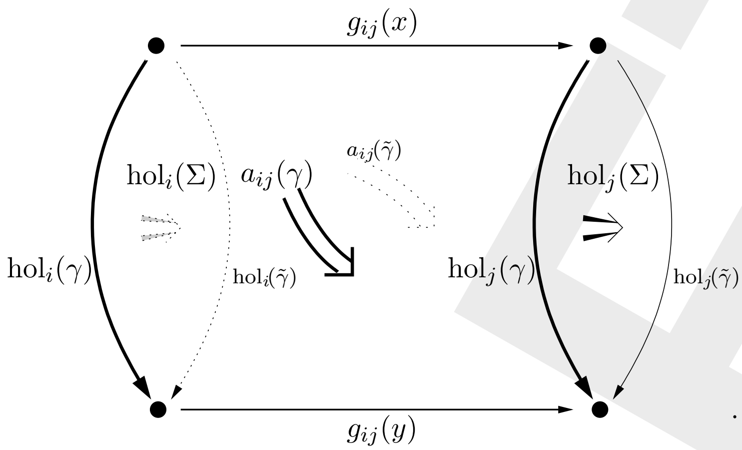
Here $\bar{t}_i$ is any one weak inverse of t_i and

$$g_{ij} \xrightarrow{\phi_{ij}} \bar{t}_i \circ t_j$$

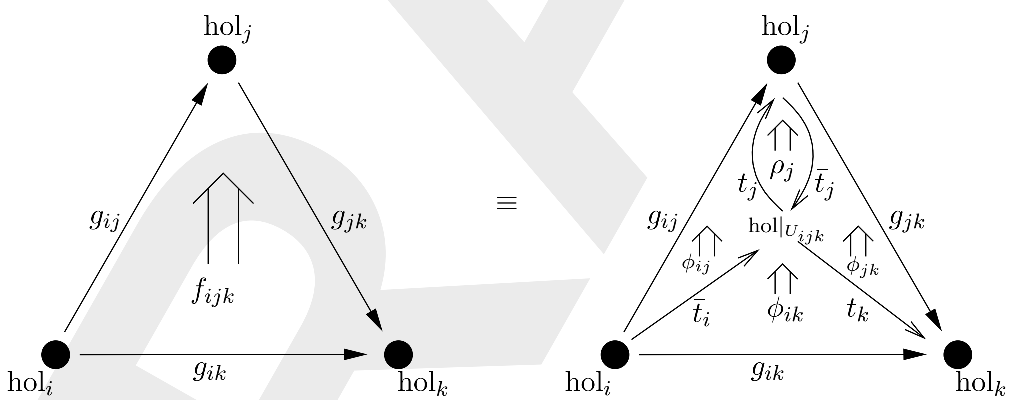
is a modification of pseudonatural transformations.

The pseudonatural transformation g_{ij} is given by a tincan diagram of this form:



. (1.3)

On triple overlaps g_{ik} and $g_{ij} \circ g_{jk}$ are related by a modification f_{ijk} which is a composite of three of the ϕ from (1.2) and of a $\rho_j \colon \mathrm{Id} \to t_j \circ \bar{t}_j$:



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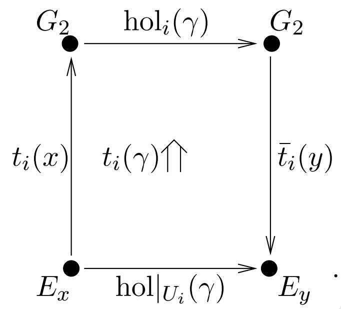
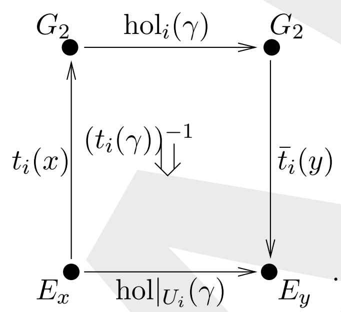
1.1.2.1 The various 2-morphisms involved. In the above diagrams we had, as usual, left all re-whiskering implicit. But for the following considerations it turns out that we need to take care of these details, lest a crucial point about the final result remains invisble.

Since we are working in the weak 2-category G_2–2Tor, there are several different things one could want to mean by "reversion" of a 2-morphism, depending on what we want to do to the source and target 1-mophisms. We shall define the following notation:

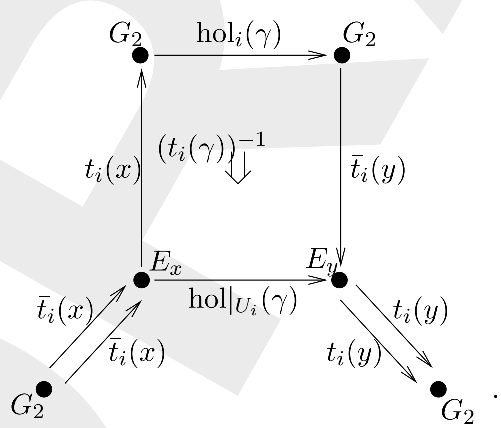
The 2-morphism in $G_2 - 2\mathrm{Tor}$ which we want to call t_i is precisely the following one:



Its inverse 2-morphism is this one:



We shall however be interested in the 2-morphism obtained from this one by reversing the 1-morphism on the left and the right. To that end we rewhisker, i.e. we horizontally compose $(t_i(\gamma))^{-1}$ with the identity 2-morphisms on $\bar{t}_i(x)$ and on $t_i(y)$:



Since $\bar{t}_i$ is only the weak inverse of t_i, we furthermore need to compose with $\eta_i \equiv \mathrm{Id} \to \bar{t}_i \circ t_i$:

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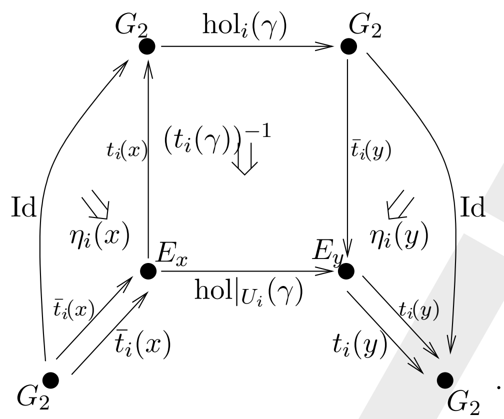
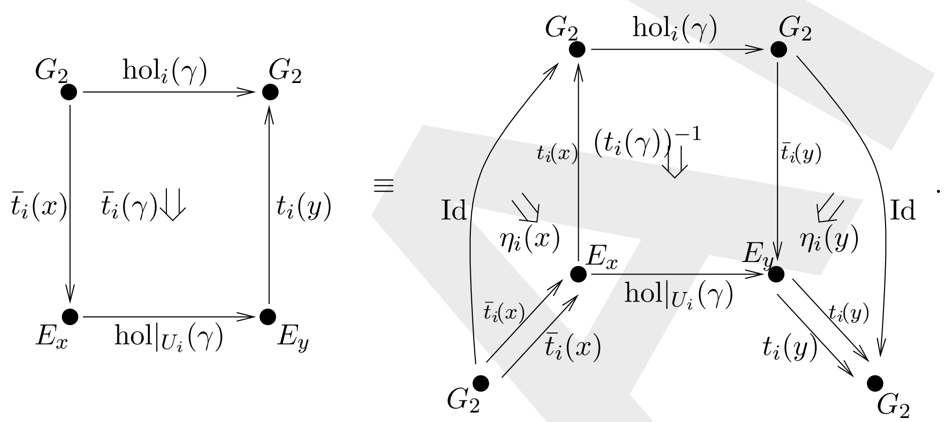


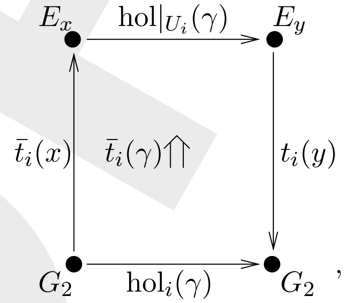
The resulting 2-morphism is the one we want to call $\bar{t}_i(\gamma)$:



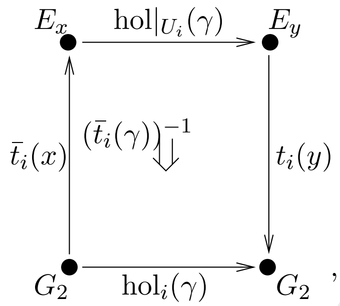
(1.4)

Below, it turns out that we need to apply this process of going from $(t_i(\gamma))^{-1}$ to $\bar{t}_i(\gamma)$ the other way around. Since this is an important step, we go through it again:

So start this time with the above 2-morphism

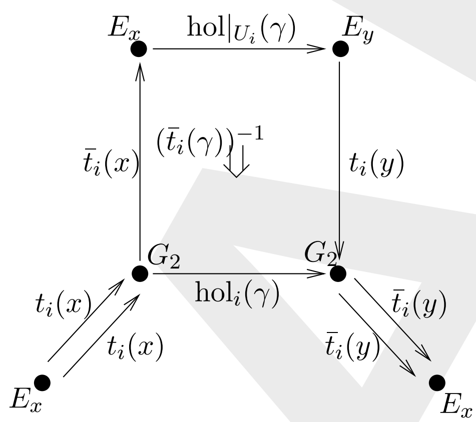


and take its inverse



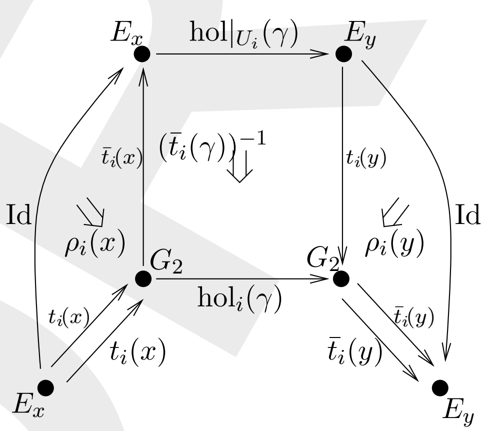
,

then re-whisker



,

and compose with $\rho_i \equiv \mathrm{Id} \to t_i \circ \bar{t}_i$



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The resulting 2-morphism must be $t_i(\gamma)$:

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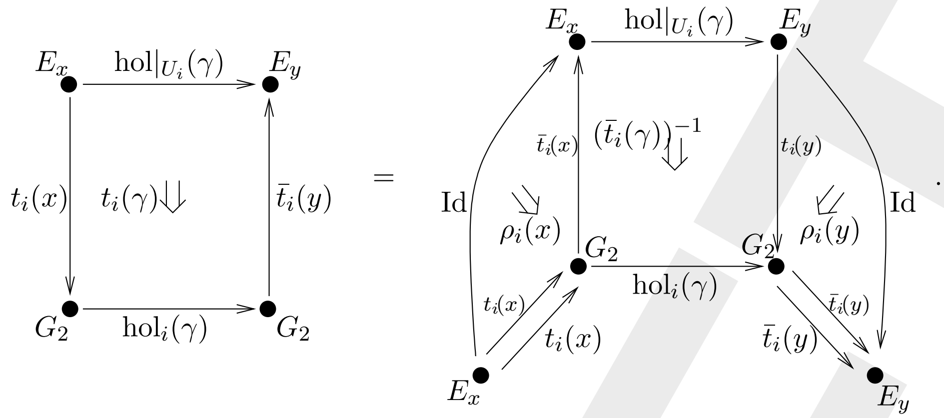


(1.5)

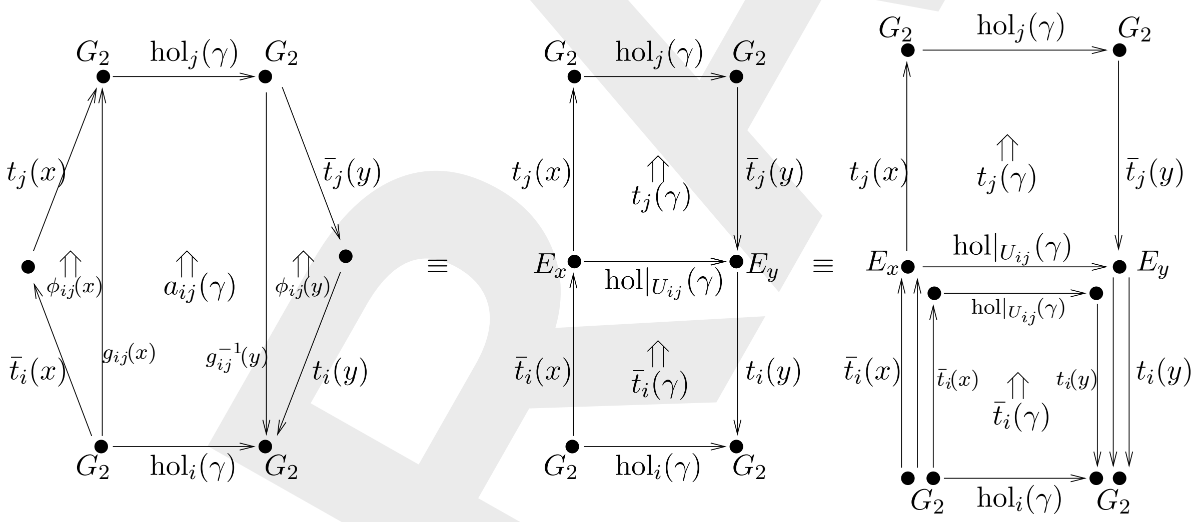
We can define the a_{ij} as the composition of $\bar{t}_i(\gamma)$ with $t_j(\gamma)$ up to modification ϕ_{ij}

$$a_{ij}(\gamma) \xrightarrow{\phi_{ij}} \bar{t}_i(\gamma) \circ t_j(\gamma)$$

as the composition $\bar{t}_i(\gamma) \circ t_j(\gamma)$ with boundary

$$g_{ij}(x) \xrightarrow{\phi_{ij}(x)} \bar{t}_i(x) \circ t_j(x) \; .$$

In order to perform this compositon we need to rewhisker t_j from the left by the identity 2-morphism $\mathrm{Id}_{\bar{t}_i}$ and from the right by the identity 2-morphism Id_{t_i}:



(1.6)

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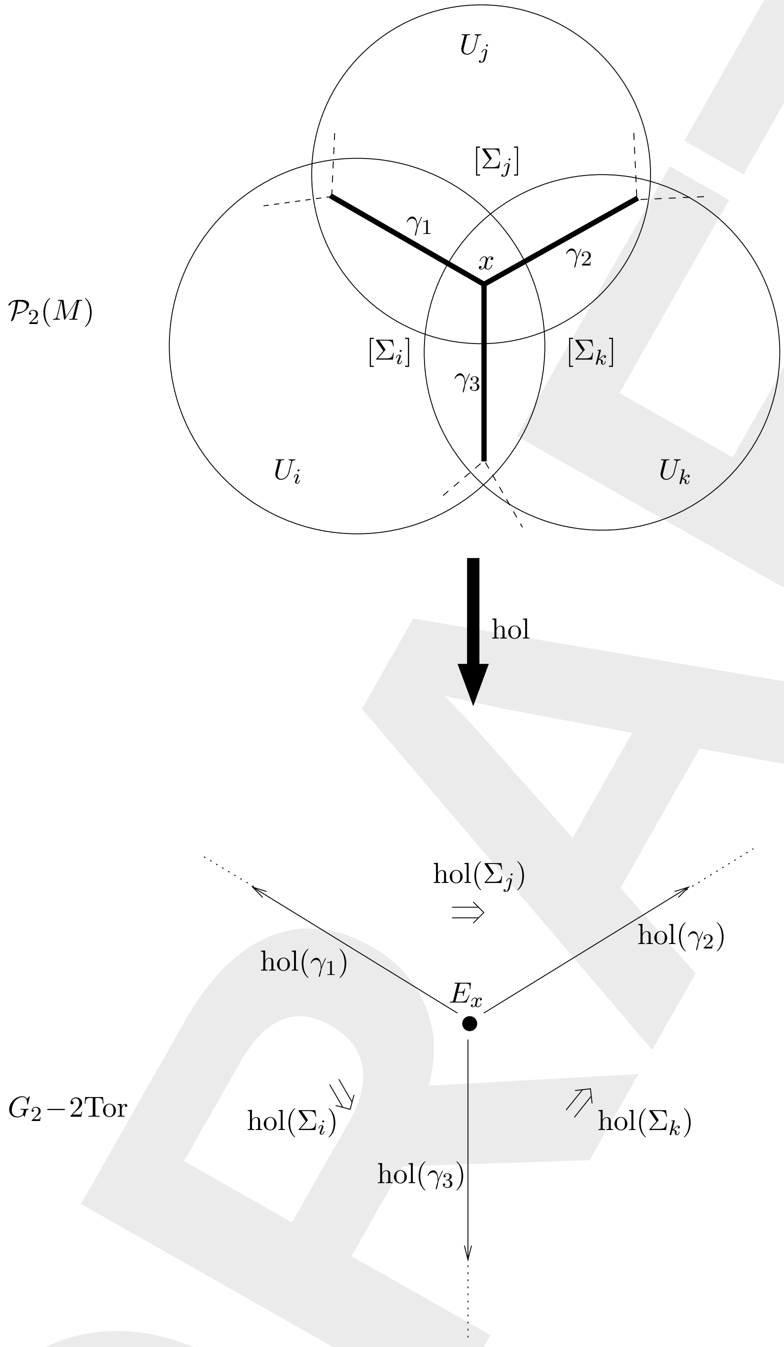
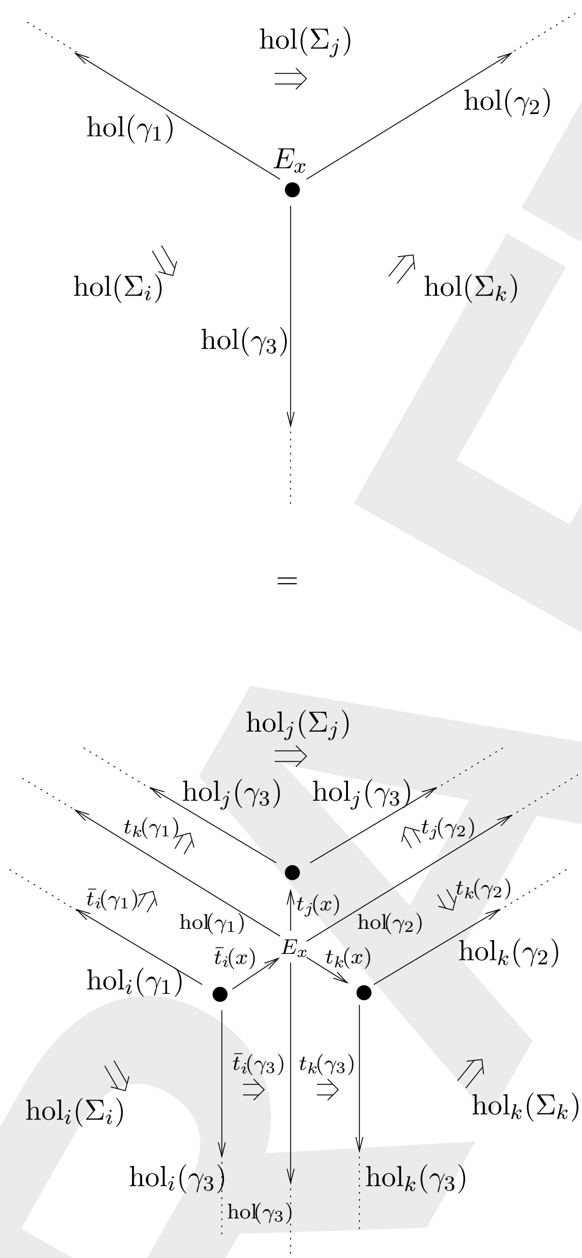
1.1.2.2 Surface Holonomy in Terms of Local Trivializations. Now consider the application of hol to any 2-morphism $[\Sigma] \in \mathcal{P}_2(M)$ that does not necessarily sit in a single patch.



We can decompose $[\Sigma]$ into several 2-morphisms that all do sit inside a single patch of the covering. Their images under hol can be regarded as the "bottom" (leftmost surface) of the tincan diagram (1.1). Since this tincan diagram 2-commutes, we can replace each $\mathrm{hol}(\Sigma_i)$ by the the respective tincan with its leftmost surface cut out. This is shown in the following figure.



We can express $t_j(\gamma_2)$ in terms of $\bar{t}_j(\gamma_2)$, using equation (1.5). This introduces the 2-morphism $\mathrm{Id} \xrightarrow{\rho_j} t_j \circ \bar{t}_j$ into the diagram:

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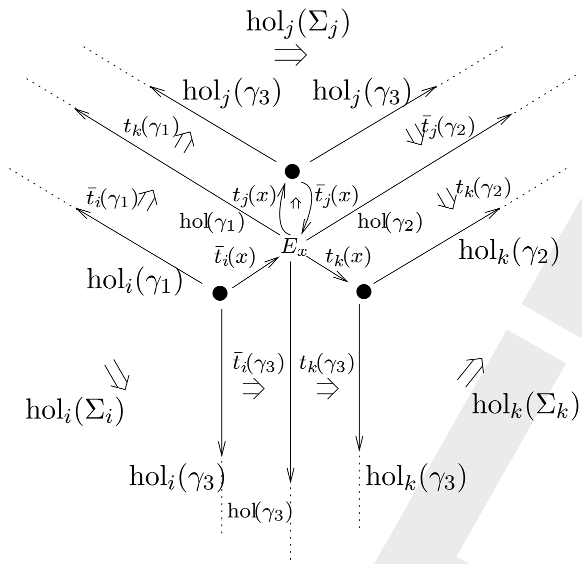
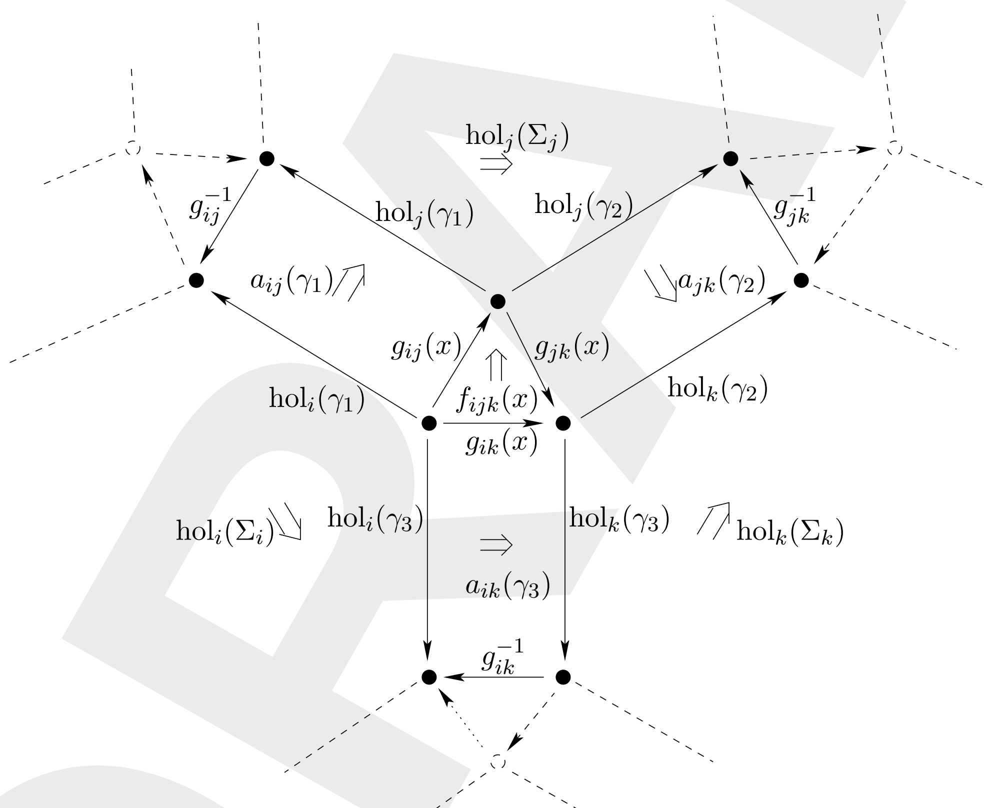


Then we can compose the t and $\bar{t}$ pairwise, using (1.6). Compare this to the discussion in §**??** (p.**??**). The result is the diagram already familiar from §**??** (p.**??**):



This shows how $\mathrm{hol}\colon \mathcal{P}_2(M) \to G_2$–2Tor can be reexpressed in terms of $\mathrm{hol}_i\colon \mathcal{P}_2(U_i) \to G$ on each U_i.

1.1.2.3 Consistency conditions. There are two consistency conditions on the t_i.

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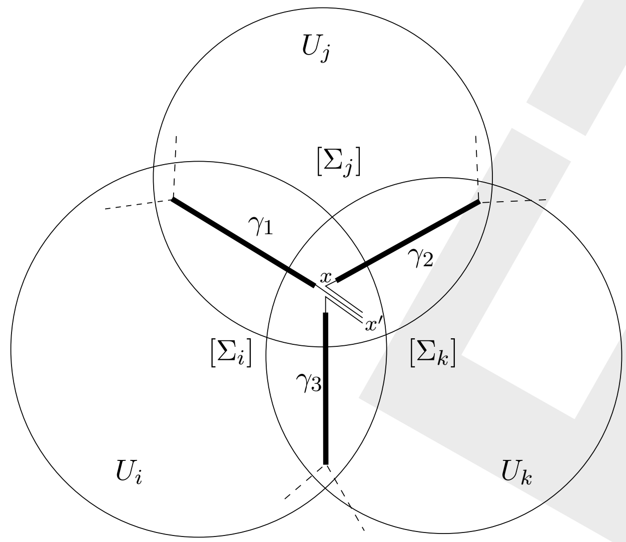
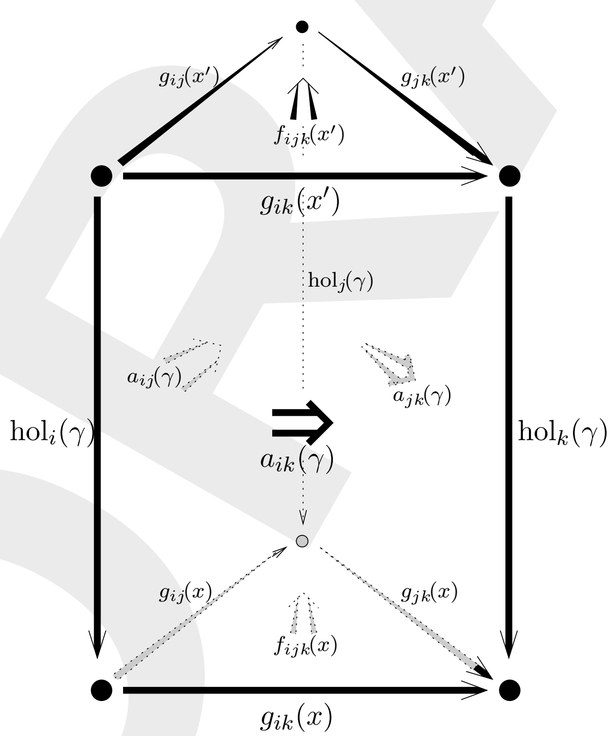
One comes from the condition that degenerate surfaces do not contribute. Consider moving $x \in U_{ijk}$ to $x' \in U_{ijk}$ by extending all of γ_1, γ_2, γ_3 by the *same* path $x \to x'$.

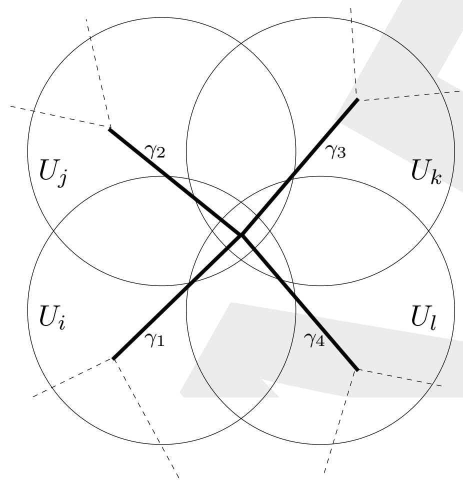


This removes $f_{ijk}(x)$ in the above diagram and replaces it by a diagram of the above form around x' but with all Σ_i vanishing. Since this must not contribute, this diagram has to equal the 2-morphism $f_{ijk}(x)$ that was replaced. In other words, the a_{ij} must be such that the following diagram 2-commutes:



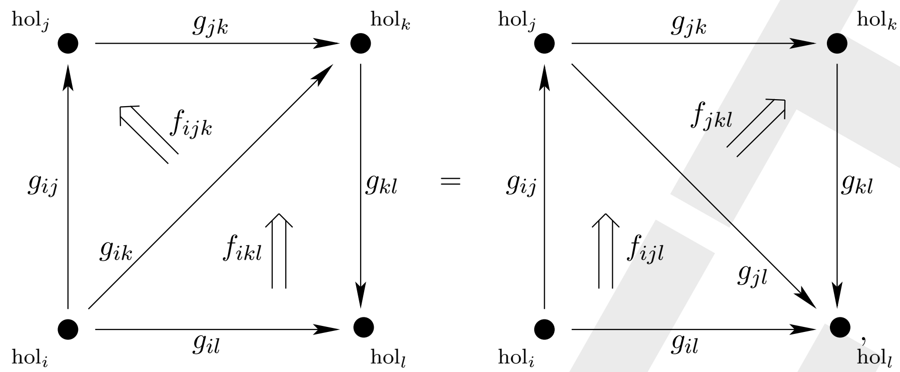
This is the transition law on triple overlaps discussed in §?? (p.??).

The other consistency condition is obtained by considering vertices at which more than three edges meet. Whenever this is the case, we can insert constant paths until only trivalent vertices are left. But these constant paths can be inserted in more than one way. Four a 4-valent vertex this is indicated by the following figure.

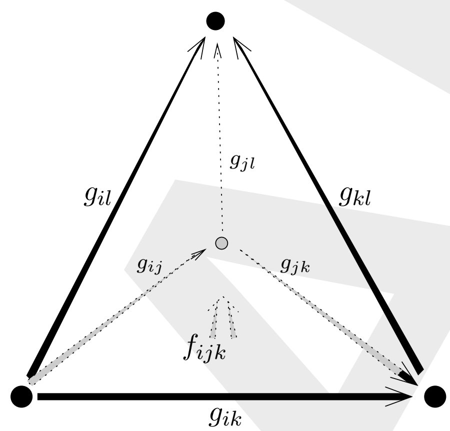


Here γ_0 denotes a constant path, which has been drawn with a spatial extension just for convenience. It is a well known theorem (see for instance [4]) that all triangulations can be obtained from any given one by a series of moves of two types, one of which is the one from one of the lower two pictures to the other. The other is the "bubble move" which was called the "left and right unit law" in §?? (p.??).

Invariance of hol under this move is expressed by the equation



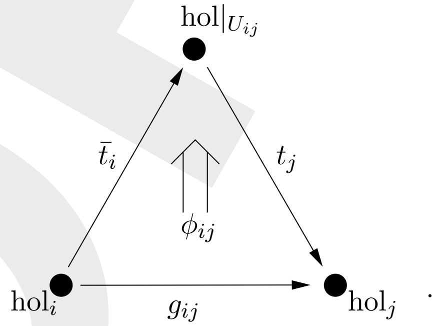
which is equivalent to the 2-commutativity of this tetrahedron:



This is the tetrahedron transition law on quadruple overlaps known from equation (**??**) (p. **??**) and from §**??** (p.**??**).

1.1.2.4 Gauge Transformations. The discussion of gauge transformations completely parallels that in §1.1.1.2 (p.6), only that now nontrivial 2-morphisms appear where previously only identity 2-morphisms were present.

Switching from one local trivialization to another corresponds to replacing the transition



(1.7)

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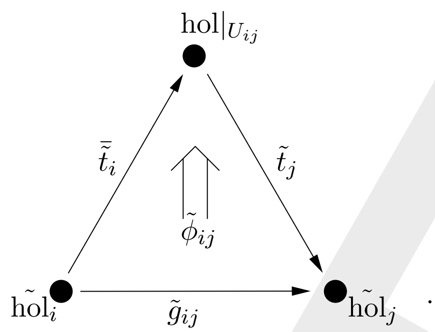
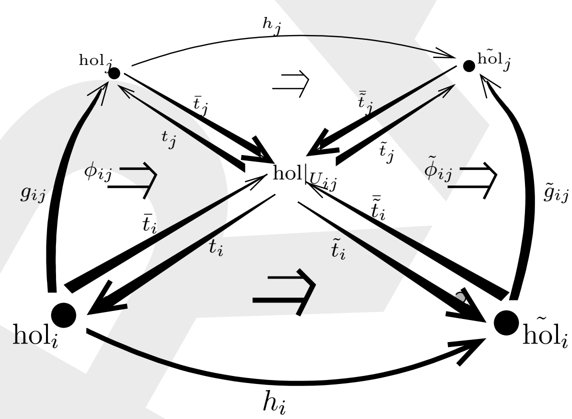
by another transition



(1.8)

Hence we get the following diagram



The existence of these modifications of pseudonatural transformations implies that a

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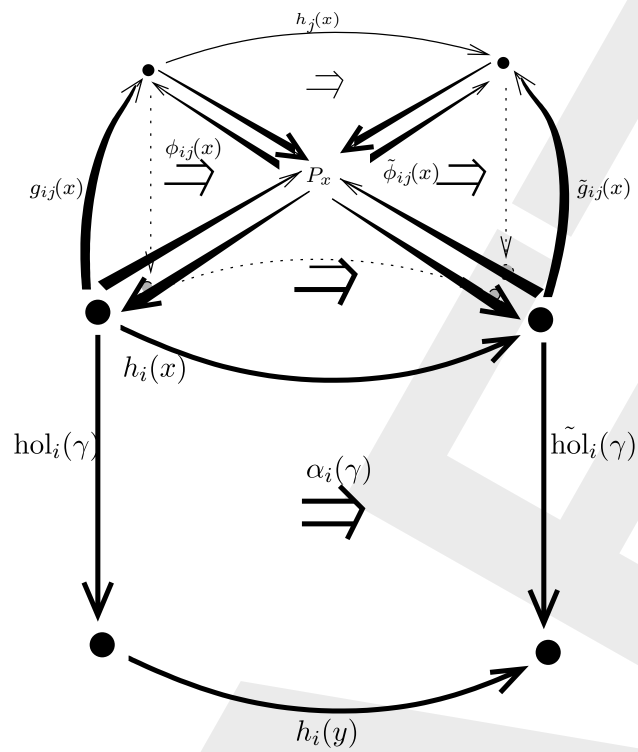
diagram as indicated in the following figure 2-commutes:



(For readability, not all 2-morphism are shown.) But this is nothing but the naturality tincan diagram describing the gauge transformations discussed in §**??** (p.**??**) This means that our 2-holonomy 2-functor

$$\mathrm{hol}\colon \mathcal{P}_2(M) \to \mathrm{Trans}_2(E)$$

encodes the same information as gauge equivalence classes of 2-functors

$$\mathrm{hol}\colon \mathcal{P}_2^C(\mathcal{U}) \to G_2$$

from the Čech-extended 2-path 2-groupoid to the structure 2-group, defined in §**??** (p.**??**).

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