

6. Canonical deformations and vertex operators

6.1 Review of first order canonical CFT deformations

It has been noted long ago [7] that adding an integrated background vertex operator V (a worldsheet field of weight (1,1)) to the string's action to first order induces a perturbation

$$L_m \rightarrow L_m + \int d\sigma e^{-im\sigma} V(\sigma) \quad (6.1)$$

of the Virasoro generators and a similar shift occurs for the supercurrent [6].

While in [7] this is discussed in CFT language it becomes quite transparent in canonical language: From the string's worldsheet action for gravitational $G_{\mu\nu}$, Kalb-Ramond $B_{\mu\nu}$ and dilaton Φ background one finds the classical stress-energy tensor (*cf.* §A (p.43))

$$T(\sigma) = \frac{1}{2} G^{\mu\nu} \frac{1}{\sqrt{2T}} \left(e^{\Phi/2} P_\mu + T \left(e^{\Phi/2} B_{\mu\kappa} + e^{-\Phi/2} G_{\mu\kappa} \right) X'^\kappa \right) \frac{1}{\sqrt{2T}} \left(e^{\Phi/2} P_\nu + T \left(e^{\Phi/2} B_{\nu\kappa} + e^{-\Phi/2} G_{\nu\kappa} \right) X'^\kappa \right) (\sigma), \quad (6.2)$$

where P_μ is the canonical momentum to X^μ .

When expanded in terms of small perturbations

$$\begin{aligned} G_{\mu\nu}(X) &= \eta_{\mu\nu} + h_{\mu\nu}(X) + \dots \\ B_{\mu\nu}(X) &= 0 + b_{\mu\nu}(X) + \dots \\ \Phi(X) &= 0 + \phi(X) + \dots \end{aligned} \quad (6.3)$$

of the background fields this yields

$$\begin{aligned} T &\approx \frac{1}{2} (\eta^{\mu\nu} - h^{\mu\nu}) \left(\mathcal{P}_{+\mu} + \sqrt{\frac{T}{2}} b_{\mu\kappa} X'^\kappa + \sqrt{\frac{T}{2}} h_{\mu\kappa} X'^\kappa + \frac{1}{\sqrt{8T}} \phi P_\mu - \sqrt{\frac{T}{8}} \phi \eta_{\mu\kappa} X'^\kappa \right) \left(\dots \right)_\nu \\ &= \frac{1}{2} \eta_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_+^\nu - \underbrace{\frac{1}{2} h_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_-^\nu}_{:=V_G} - \underbrace{\frac{1}{2} b_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_-^\nu}_{:=V_B} + \underbrace{\frac{1}{2} \phi \eta_{\mu\nu} \mathcal{P}_+^\mu \mathcal{P}_-^\nu}_{:=V_\Phi} + \text{higher order terms}, \end{aligned} \quad (6.4)$$

where we have defined

$$\mathcal{P}_\pm^\mu(\sigma) := \frac{1}{\sqrt{2T}} (\eta^{\mu\nu} P_\nu \pm T X'^\nu)(\sigma). \quad (6.5)$$

It must be noted that while the objects \mathcal{P}_\pm , which have Poisson-bracket

$$\{\mathcal{P}_\pm^\mu(\sigma), \mathcal{P}_\pm^\nu(\sigma')\} = \mp \eta^{\mu\nu} \delta'(\sigma - \sigma'), \quad (6.6)$$

generate the current algebra of the free theory, they involve, via $P_\mu = \delta S / \delta \dot{X}^\mu$, data of the perturbed background and are hence not proportional to ∂X and $\bar{\partial} X$.

Still, the first term in (6.4) is the generator of the Virasoro algebra which is associated with the U(1)-currents \mathcal{P}_\pm , while the following terms are the weight (1,1) vertices V_G , V_B , V_Φ (with respect to the first term) of the graviton, 2-form and dilaton, respectively.

Hence in the sense that we regard the canonical coordinates and momenta as fundamental and hence unaffected by the background perturbation, i.e.

$$\begin{aligned} X^\mu &\rightarrow X^\mu \\ P_\mu &\rightarrow P_\mu, \end{aligned} \tag{6.7}$$

while only the ‘coupling constants’ are shifted

$$\eta_{\mu\nu} \rightarrow \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{etc.} \tag{6.8}$$

we can write

$$T \rightarrow T + V, \tag{6.9}$$

where V denotes a collection of weight (1,1) vertices in the above sense.⁴

CFT deformations of this form are called *canonical deformations* [8, 27].

The central idea of canonical first order deformations is that the (super-) Virasoro algebra

$$\begin{aligned} [T(\sigma), T(\sigma')] &= 2iT(\sigma') \delta'(\sigma - \sigma') - iT'(\sigma') \delta(\sigma - \sigma') + A(\sigma - \sigma') \\ \{T_F(\sigma), T_F(\sigma')\} &= -\frac{1}{2\sqrt{2}}T(\sigma') \delta(\sigma') + B(\sigma - \sigma') \\ [T(\sigma), T_F(\sigma')] &= \frac{3i}{2}T_F(\sigma') \delta'(\sigma - \sigma') - iT'_F(\sigma') \delta(\sigma - \sigma') \end{aligned} \tag{6.10}$$

(where A and B are the anomaly terms) together with its chiral partner, generated by \bar{T} and \bar{T}_F , is preserved to first order under the perturbation

$$\begin{aligned} T(\sigma) &\rightarrow T(\sigma) + \delta T(\sigma) \\ T_F(\sigma) &\rightarrow T_F(\sigma) + \delta T_F(\sigma) \end{aligned} \tag{6.11}$$

if, in particular,

$$\begin{aligned} \delta T(\sigma) &= \Phi(\sigma) \bar{\Phi}(\sigma) \\ \delta F(\sigma) &= \Phi_F(\sigma) \bar{\Phi}_F(\sigma) \end{aligned} \tag{6.12}$$

with

$$\begin{aligned} [T(\sigma), \Phi(\sigma')] &= i\Phi(\sigma') \delta'(\sigma - \sigma') - i\Phi'(\sigma') \delta(\sigma - \sigma') \\ [T(\sigma), \Phi_F(\sigma')] &= \frac{i}{2}\Phi(\sigma') \delta'(\sigma - \sigma') - i\Phi'(\sigma') \delta(\sigma - \sigma') \\ [T(\sigma), \bar{\Phi}(\sigma')] &= 0 \\ [T(\sigma), \bar{\Phi}_F(\sigma')] &= 0 \end{aligned} \tag{6.13}$$

⁴As is discussed in [7], the issue concerning (6.7) in CFT language translates into the question whether one chooses to treat ∂X and $\bar{\partial} X$ as free fields in the perturbed theory and whether the $\partial X \partial X$ -OPE is taken to receive a perturbation or not.

For a further discussion of perturbations of SCFTs where this issue is addressed, see [4] and in particular section 2.2.4.

and analogous relations for $\delta\bar{T}$ and $\delta\bar{T}_F$.

There are however also more general fields δT , δT_F of total weight 2 and 3/2, respectively, which preserve the above super-Virasoro algebra to first order [28]. But the weight (1,1) part $\Phi(\sigma)\bar{\Phi}(\sigma)$ is special in that it corresponds directly to the vertex operator of the background which is described by the deformed worldsheet theory. Further deformation fields of weight different from (1,1) are related to *gauge* degrees of freedom of the background fields (*cf.* [28] and the discussion below equation (6.24)).

6.2 Canonical deformations from $\mathbf{d}_K \rightarrow e^{-\mathbf{W}}\mathbf{d}_K e^{\mathbf{W}}$.

We would like to see how the deformation theory reviewed above relates to the SCFT deformations that have been studied in §3 (p.12).

First recall from §3.1 (p.12) that the chiral bosonic fields ∂X and $\bar{\partial} X$ in our notation read

$$\mathcal{P}_{\pm}(\sigma) := \frac{1}{\sqrt{2T}} \left(-i \frac{\delta}{\delta X} \pm T X' \right) (\sigma) \quad (6.14)$$

and that according to §2.2.2 (p.9) we write the worldsheet fermions ψ , $\bar{\psi}$ as Γ_{\pm} , respectively, which are normalized so that $\{\Gamma_{\pm}^{\mu}(\sigma), \Gamma_{\pm}^{\nu}(\sigma')\} = \pm 2g^{\mu\nu}(X(\sigma)) \delta(\sigma - \sigma')$, and we frequently make use of the linear combinations

$$\begin{aligned} \mathcal{E}^{\dagger\mu} &= \frac{1}{2} (\Gamma_+^{\mu} + \Gamma_-^{\mu}) \\ \mathcal{E}^{\mu} &= \frac{1}{2} (\Gamma_+^{\mu} - \Gamma_-^{\mu}) . \end{aligned} \quad (6.15)$$

In this notation the supercurrents for the trivial background read

$$\begin{aligned} T_F(\sigma) &= \frac{1}{\sqrt{2}} \Gamma_+(\sigma) \mathcal{P}_+(\sigma) = \frac{-i}{\sqrt{4T}} \left(\mathbf{d}_K(\sigma) - \mathbf{d}^{\dagger}_K(\sigma) \right) \\ \bar{T}_F(\sigma) &= \frac{i}{\sqrt{2}} \Gamma_-(\sigma) \mathcal{P}_-(\sigma) = \frac{1}{\sqrt{4T}} \left(\mathbf{d}_K(\sigma) + \mathbf{d}^{\dagger}_K(\sigma) \right) , \end{aligned} \quad (6.16)$$

where the K -deformed exterior derivative and coderivative on loop space are identified as

$$\begin{aligned} \mathbf{d}_K &= \sqrt{T} (\bar{T}_F + iT_F) \\ \mathbf{d}^{\dagger}_K &= \sqrt{T} (\bar{T}_F - iT_F) . \end{aligned} \quad (6.17)$$

According to §3.2 (p.15) a consistent deformation of the superconformal algebra generated by T_F and \bar{T}_F is given by sending

$$\begin{aligned} \mathbf{d}_K(\sigma) &\rightarrow \mathbf{d}_K^{(W)}(\sigma) = e^{-\mathbf{W}} \mathbf{d}_K(\sigma) e^{\mathbf{W}} = \mathbf{d}_K(\sigma) + \underbrace{[\mathbf{d}_K(\sigma), \mathbf{W}]}_{:=\delta\mathbf{d}_K(\sigma)} + \dots \\ \mathbf{d}^{\dagger}_K(\sigma) &\rightarrow \mathbf{d}^{\dagger}_K^{(W)}(\sigma) = e^{\mathbf{W}^{\dagger}} \mathbf{d}^{\dagger}_K(\sigma) e^{-\mathbf{W}^{\dagger}} = \mathbf{d}^{\dagger}_K(\sigma) + \underbrace{[\mathbf{W}^{\dagger}, \mathbf{d}^{\dagger}_K(\sigma)]}_{:=\delta\mathbf{d}^{\dagger}_K(\sigma)} + \dots \end{aligned} \quad (6.18)$$

for \mathbf{W} some reparameterization invariant operator. From this one finds δT_F by using (6.16)

$$\begin{aligned}\delta T_F &= -i \left[\bar{T}_F, \frac{1}{2} (\mathbf{W} + \mathbf{W}^\dagger) \right] + \left[T_F, \frac{1}{2} (\mathbf{W} - \mathbf{W}^\dagger) \right] \\ \delta \bar{T}_F &= i \left[T_F, \frac{1}{2} (\mathbf{W} + \mathbf{W}^\dagger) \right] + \left[\bar{T}_F, \frac{1}{2} (\mathbf{W} - \mathbf{W}^\dagger) \right]\end{aligned}\quad (6.19)$$

which again gives δT by means of

$$\{T_F(\sigma), \delta T_F(\sigma')\} + \{T_F(\sigma'), \delta T_F(\sigma)\} = -\frac{1}{2\sqrt{2}} \delta T(\sigma) \delta(\sigma - \sigma') . \quad (6.20)$$

Before looking at special cases one should note that this necessarily implies that δT_F is of total weight $3/2$ and that δT is of total weight 2 . That's because \mathbf{W} , being reparameterization invariant, must be the integral (along the string at fixed worldsheet time) over a field of unit weight (*cf.* (3.20)) and because supercommutation with \mathbf{d}_K or \mathbf{d}^\dagger_K increases the total weight by $1/2$.

Furthermore, recall from (3.25) that the anti-hermitean part $\frac{1}{2} (\mathbf{W} - \mathbf{W}^\dagger)$ of the deformation operator \mathbf{W} is responsible for *pure gauge transformations* while the hermitean part $\frac{1}{2} (\mathbf{W} + \mathbf{W}^\dagger)$ induces true modifications of the background fields. Hence for a pure gauge transformation (6.19) yields

$$\begin{aligned}\delta T_F &= [T_F, \mathbf{W}] \\ \delta \bar{T}_F &= [\bar{T}_F, \mathbf{W}] , \quad \text{for } \mathbf{W}^\dagger = -\mathbf{W} ,\end{aligned}\quad (6.21)$$

which of course comes from the global similarity transformation (3.25)

$$\mathbf{X} \mapsto e^{-\mathbf{W}} \mathbf{X} e^{\mathbf{W}} , \quad \mathbf{X} \in \{T_F, \bar{T}_F, \dots\} . \quad (6.22)$$

On the other hand, for a strictly non-gauge transformation the transformation (6.19) simplifies to

$$\begin{aligned}\delta T_F &= -i [\bar{T}_F, \mathbf{W}] \\ \delta \bar{T}_F &= i [T_F, \mathbf{W}] , \quad \text{for } \mathbf{W}^\dagger = +\mathbf{W} .\end{aligned}\quad (6.23)$$

In the cases where \mathbf{W} is antihermitean *and* a $(1/2, 1/2)$ field (as is in particular the case for the gravitational $\mathbf{W}^{(G)}$ of §3.3 (p.16), the dilaton $\mathbf{W}^{(D)}$ of §3.5 (p.18) and the hermitean part of the Kalb-Ramond $\mathbf{W}^{(B)} + \mathbf{W}^{(B)\dagger}$ of §3.4 (p.17)) this, together with (6.20) implies that

$$\delta T \propto \{T_F, [\bar{T}_F, \mathbf{W}]\} \quad (6.24)$$

is indeed of weight $(1, 1)$, as discussed in the theory of canonical deformations §6.1 (p.36). Furthermore, this shows explicitly that all contributions to δT which are of total weight 2 but *not* of weight $(1, 1)$ must come from the antihermitean component $\frac{1}{2} (\mathbf{W} - \mathbf{W}^\dagger)$ and hence must be associated with background gauge transformations. This proves in full generality the respective observation in [28] concerning 2-form field deformations.

As a first example, consider the deformation induced by a B -field background:

B-field background. According to §3.4 (p.17) a Kalb-Ramond background is induced by choosing

$$\mathbf{W} = \int d\sigma \frac{1}{2} B_{\mu\nu}(X(\sigma)) \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu} \quad (6.25)$$

which, using (6.18) gives rise to

$$\begin{aligned} \delta \mathbf{d}_K(\sigma) &= \left(\frac{1}{6} H_{\alpha\beta\gamma}(X) \mathcal{E}^{\dagger\alpha} \mathcal{E}^{\dagger\beta} \mathcal{E}^{\dagger\gamma} - iT \mathcal{E}^{\dagger\mu} B_{\mu\nu}(X) X^{\nu} \right) (\sigma) \\ \delta \mathbf{d}^{\dagger}_K(\sigma) &= \left(-\frac{1}{6} H_{\alpha\beta\gamma}(X) \mathcal{E}^{\alpha} \mathcal{E}^{\beta} \mathcal{E}^{\gamma} + iT \mathcal{E}^{\mu} B_{\mu\nu}(X) X^{\nu} \right) (\sigma) \end{aligned} \quad (6.26)$$

and hence, using (6.16), to

$$\delta T_F(\sigma) = -\frac{i}{12\sqrt{T}} H_{\alpha\beta\gamma} \left(\mathcal{E}^{\dagger\alpha} \mathcal{E}^{\dagger\beta} \mathcal{E}^{\dagger\gamma} + \mathcal{E}^{\alpha} \mathcal{E}^{\beta} \mathcal{E}^{\gamma} \right) - \frac{1}{\sqrt{8}} \Gamma_+^{\mu} B_{\mu\nu} (\mathcal{P}_+^{\nu} - \mathcal{P}_-^{\nu}) . \quad (6.27)$$

In this special case δT_F happens to be the exact shift of T_F (there are no higher order perturbations of T_F in this background). As has been noted already in §3.4 (p.17) the same result is obtained by canonically quantizing the supersymmetric $2d$ σ -model (3.38) which describes superstrings in a Kalb-Ramond background.

By means of (6.20) the shift δT is easily found to be

$$\begin{aligned} \delta T(\sigma) &= \left(-\frac{1}{12T} \partial_{\delta} H_{\alpha\beta\gamma} \left(\mathcal{E}^{\delta} \mathcal{E}^{\dagger\alpha} \mathcal{E}^{\dagger\beta} \mathcal{E}^{\dagger\gamma} + \mathcal{E}^{\dagger\delta} \mathcal{E}^{\alpha} \mathcal{E}^{\beta} \mathcal{E}^{\gamma} \right) - i \frac{1}{2\sqrt{2T}} H_{\alpha\beta\gamma} \left(\mathcal{E}^{\dagger\alpha} \mathcal{E}^{\dagger\beta} + \mathcal{E}^{\alpha} \mathcal{E}^{\beta} \right) \mathcal{P}_+^{\gamma} \right. \\ &\quad \left. + \frac{i}{\sqrt{4T}} \partial_{\delta} B_{\mu\nu} (\mathcal{P}_+^{\nu} - \mathcal{P}_-^{\nu}) \Gamma_+^{\delta} \Gamma_+^{\mu} + B_{\mu\nu} \mathcal{P}_+^{\mu} \mathcal{P}_-^{\nu} \right) (\sigma) . \end{aligned} \quad (6.28)$$

This is of total weight 2 and contains the weight (1,1) vertex operator

$$V = B_{\mu\nu} \mathcal{P}_+^{\mu} \mathcal{P}_-^{\nu} \quad (6.29)$$

of the Kalb-Ramond field (*cf.* eqs. (52),(53) in [28]). That $T + \delta T$ satisfies the Virasoro algebra to first order at the level of Poisson brackets follows from the fact that it derives from a consistent deformation of the form (3.22) (as well as from the fact that it also derives from the respective σ -model Lagrangian). When anomalies of the quantum algebra are taken into account their vanishing are equivalent to the equations of motion of the B -field to first order (see [28] and §5 (p.32)).