

Holonomy on Smooth Path Spaces

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Abstract

This are private unfinished notes concerning the issue of holonomy on path spaces. I try to give a rigorous proof that every smooth local 2-holonomy 2-functor comes from a fake flat pair of a 1-form A and a 2-form B .

Two approaches are examined. First I try to show that for *any* smooth space X a thin-invariant group-valued 0-form on the path space PX (a holonomy on X) is given by a 1-form (in the sense of Chen) on X . This generalizes the known case where $X = M$ is an ordinary manifold to cases like $X = PM$. The entire discussion is an exercise in Chen's theory of smooth spaces and parallel transport on them.

This should show that every 2-holonomy 2-functor comes from a 1-form on path space. It does, however, probably not show that this 1-form has to come from pulling back a 2-form on target space to paths.

I note that ideas from synthetic differential geometry seem to be ideally suited to prove things like that. In the second part I sketch what should be a proof that every 2-holonomy 2-functor gives rise to a 2-form on target space. The idea is that such a functor easily gives rise to a *combinatorial* 2-form in the sense of synthetic differential geometry. It turns out that for this to work we need precisely the fact that the 2-functor is smooth in the sense of Chen and is thin-invariant.

Contents

1	Smooth Spaces	3
1.1	Introduction	3
1.2	Basic Definitions	3
1.3	Structures on Smooth Spaces	4
1.3.1	p -Forms	5
1.4	Path Spaces	5
1.4.1	Basic Definitions	5
1.4.2	Iterated Integrals	7
1.5	Holonomy on Smooth Spaces	7
2	Synthetic Differential Geometry	13
2.1	Synthetic Differential Geometry in Terms of Infinitesimals .	13
2.1.1	Exponentials and Logarithms	16
2.2	Combinatorial Differential Forms for Mortals	16
2.3	p -Forms from p -Holonomy Functors	18
2.3.1	Simplices as p -Morphisms	19
2.3.2	1-Connections	20
2.3.3	2-Connections	20

1 Smooth Spaces

1.1 Introduction

In [10] it was assumed without proof that a holonomy 2-functor on a space of paths of paths in X comes from a Chen-1-form on the space of paths in X , just like an ordinary holonomy functor on paths in X comes from an ordinary 1-form on X . In private communication Hendryk Pfeiffer pointed out that such an assumption is not self-evident.

The following is a draft version of what is supposed to be a rigorous proof of this assumption.

First some basic material on Chen's theory of smooth spaces and iterated integrals is reviewed and the '*path functor*' is made explicit. Then the relation between holonomy and 1-forms for ordinary manifolds is recalled. By applying it on each plot of a smooth space it is proven that the statement generalizes to arbitrary smooth spaces and Chen-forms on them. This is the content of prop. 3 on p. 8.

1.2 Basic Definitions

The following collects and reviews some basic definitions and facts concerning smooth spaces as given by Chen. See for instance the old review paper [1] which can be found reproduced in [2]. See also the appendix in the revised version of [10]. A nice modern account of the basic ideas of smooth spaces (also known as the theory of *diffeology*) relating it to sheaf and topos theory is given in section 1 of [7].

A *smooth space* is a special case of a *differentiable space*. A differentiable space is a set which is equipped with a possibly much weaker (and much more general) sense of differentiability than an ordinary manifold.

The idea is to access a given set X only by means of maps $U \xrightarrow{\phi} X$, called *plots*, from subsets U of \mathbb{R}^n for *all* n .

The choice of plots determines how exotic the notion of differentiability on X will look like. Well-behaved plots are for instance continuous, in which case the differentiable space is called a *smooth space*.

More in detail, plots are defined as follows:

Definition 1 *Let X be some set. A category of plots of X is a category pl whose objects are set maps*

$$\text{Obj}(\text{pl}) = \left\{ U_i \xrightarrow{\phi_i} X \right\}$$

for $\{U_i \subset \mathbb{R}^{n_i}\}_{i \in I}$ a collection of open convex subsets of real vector spaces and including all constant set maps, with one morphism $\phi_i \xrightarrow{\Lambda} \phi_j$ for every $C^\infty \ni \lambda: U_i \rightarrow U_j$ given by

$$\phi_i \xrightarrow{\Lambda} \phi_j \quad \Leftrightarrow \quad \begin{array}{ccc} U_i & \xrightarrow{\lambda} & U_j \\ \phi_i \downarrow & & \downarrow \phi_j \\ X & = & X \end{array}$$

and endowed with the obvious composition operation.

By equipping sets with plots we obtain various flavors of differentiable spaces:

Definition 2

- A **predifferentiable space** (X, pl_X) is a set X together with a category pl_X of plots of X .
- A **differentiable space** (X, pl_X) is a predifferentiable space with the additional property that given any set map $U \xrightarrow{\phi} X$ such that there is a covering of U by sets U_i and such that each $\phi|_{U_i} \in \text{Obj}(\text{pl}_X)$, then $\phi \in \text{Obj}(\text{pl}_X)$.
- A **smooth space** (or **topological differentiable space**) is a differentiable space (X, pl_X) together with a topology on X such that all plots in pl_X are continuous.

Remark 1 Every predifferentiable space gives rise to an associated differentiable space in a natural way.

Definition 3 The **product** of two differentiable spaces (X, pl_X) and (Y, pl_Y) is the differentiable space

$$(X, \text{pl}_X) \times (Y, \text{pl}_Y) = (X \times Y, \text{pl}_X \times \text{pl}_Y).$$

The product of two differentiable maps $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ is

$$X \times X' \xrightarrow{f \times f'} Y \times Y'.$$

Example 1

1. An ordinary manifold M is a differentiable space (M, pl_M) with plots in pl_M being the inverses of the charts of M .

A morphism between differentiable spaces is called a differentiable map:

Definition 4 A **differentiable map** (smooth map)

$$(X, \text{pl}_X) \xrightarrow{F} (Y, \text{pl}_Y)$$

between differentiable (smooth) spaces is a set map (continuous map) $X \xrightarrow{F} Y$ such that

$$\begin{array}{ccc} U_i & = & U_i \\ \text{pl}_X \ni \phi_i \downarrow & & \downarrow \phi_i \circ F \in \text{pl}_Y \\ X & \xrightarrow{F} & Y \end{array}$$

for all $\phi_i \in \text{pl}_X$.

1.3 Structures on Smooth Spaces

Some constructions for ordinary manifolds carry over to differentiable spaces by applying them on each plot separately in a way that respects the morphisms between plots. In particular, a p -form on a differentiable space (X, pl_X) can be defined as a form on each plot in pl_X such that plot morphisms correspond to pullbacks.

1.3.1 p -Forms

Definition 5 A p -form ω on a differentiable space (X, pl_X) is an assignment of an ordinary p -form

$$\omega : \text{pl}_X \ni \phi_i \mapsto \phi_i^* \omega \in \Omega^p(U_i)$$

on the domain of each plot $U_i \xrightarrow{\phi_i} X$ such that

$$\begin{array}{ccccc} U_i & \xrightarrow{\lambda} & U_j & \xrightarrow{\phi_j} & X \\ \downarrow & & \downarrow & & \downarrow \\ \phi_i^* \omega & \xleftarrow{\lambda^*} & \phi_j^* \omega & \xleftarrow{\phi_j^*} & \omega \end{array}$$

for all $i \in I$.

Note that here λ^* is the ordinary pullback of ordinary forms for $\lambda \in C^\infty(U_i, U_j)$, while $\phi^* \omega$ for $\phi \in \text{pl}_X$ is new primitive notation.

The space of p -forms on (X, pl_X) is denoted by $\Omega^p((X, \text{pl}_X))$. The above definition immediately generalizes for instance to Lie group valued 0-forms $\Omega^0((X, \text{pl}_X), G)$ and to Lie algebra valued p -forms $\Omega^p((X, \text{pl}_X), \text{Lie}(G))$. An element $F \in \Omega^0((X, \text{pl}_X), G)$ is the same as a smooth map $(X, \text{pl}_X) \xrightarrow{F} (G, \text{pl}_G)$ with the Lie group G regarded as a smooth space.

Definition 6 Let

$$\begin{array}{ccc} (X, \text{pl}_X) & \xrightarrow{F} & (Y, \text{pl}_Y) \\ X & \xrightarrow{f} & Y \end{array}$$

be a smooth map and let $\omega \in \Omega^p((Y, \text{pl}_Y))$. Then there is a **pullback form** $F^* \omega \in \Omega^p((X, \text{pl}_X))$ defined by

$$F^* \omega : \phi \mapsto (\phi \circ f)^* \omega, \quad \forall \phi \in \text{Obj}(\text{pl}_X).$$

Note that this is well defined because f being a differentiable map ensures that $U \xrightarrow{\phi} X \xrightarrow{f} Y \in \text{Obj}(\text{pl}_Y)$.

1.4 Path Spaces

1.4.1 Basic Definitions

The path space over some target space is the space of smooth maps from the interval $I = [0, 1]$ into the target space.

Definition 7 The **path space** (PX, pl_{PX}) of a smooth space (X, pl_X) is the smooth space such that

1. PX is the set of smooth maps $I \xrightarrow{\gamma} X$ (equipped with the compact-open topology?), where we write evaluation of paths as

$$PX \times I \xrightarrow{\text{ev}} X,$$

2. pl_{PX} contains precisely all those maps $U_i \xrightarrow{\phi_i} PX$ such that

$$U_i \times I \xrightarrow{\phi_i \times \text{Id}} PX \times I \xrightarrow{\text{ev}} X$$

is a plot of X .

Note that this choice of plots in particular makes $PX \times I \xrightarrow{\text{ev}} X$ a smooth map.

Definition 8 Given any smooth map $X \xrightarrow{f} Y$ and the path spaces PX and PY the map $PX \xrightarrow{Pf} PY$ is the map defined by

$$\begin{array}{ccc} PX \times I & \xrightarrow{Pf \times \text{Id}_I} & PY \times I \\ \text{ev} \downarrow & & \downarrow \text{ev} \\ X & \xrightarrow{f} & Y \end{array} .$$

Proposition 1 The map Pf defined as above is unique and in fact a smooth map.

Proof. Uniqueness follows from the fact that the diagram implies that $\text{ev}|_{Pf(\gamma)}(\sigma) = f(\text{ev}|_\gamma(\sigma))$.

To see that Pf is a smooth map consider any plot $U \xrightarrow{\phi} PX$ in pl_{PX} . Since ev and f are smooth maps it follows that

$$U \times I \xrightarrow{\phi \times \text{Id}} PX \times I \xrightarrow{\text{ev}} X \xrightarrow{f} Y$$

is a plot in pl_{PY} . But by the definition of Pf this equals

$$U \times I \xrightarrow{\phi \times \text{Id}} PX \times I \xrightarrow{Pf \times \text{Id}} PY \times I \xrightarrow{\text{ev}} X \xrightarrow{f} Y .$$

By point 2 in the definition of path spaces this implies that

$$U \xrightarrow{\phi} PX \xrightarrow{Pf} PY$$

is a plot in pl_{PY} . Since ϕ was arbitrary this shows that Pf is smooth. \square

It follows that P is in fact a functor, the **path functor** on the category of smooth spaces

$$\begin{array}{ccc} P & : & \text{Diff} \rightarrow \text{Diff} \\ & & X \quad \quad PX \\ & & f \downarrow \quad \mapsto \quad \downarrow Pf \\ & & Y \quad \quad PY \end{array} .$$

In the context of holonomy one is interested in 0-forms on path space which take the same value on paths related by thin homotopy. As usual, we can generalize this concept from path spaces over manifolds to arbitrary path spaces by applying it plotwise:

Definition 9

1. A 0-form $\Omega^0((PM, \text{pl}_{PM}))$ on the path space PM over a manifold M is called **thin invariant** if it takes the same value on paths which are related by thin homotopy.
2. A 0-form $f \in \Omega^0((PX, \text{pl}_{PX}))$ on the path space PX over a smooth space X is called **thin invariant** if, for every plot $U \xrightarrow{\phi} X$ and its lift $PU \xrightarrow{P\phi} PX$ to path space, the pullback

$$(P\phi)^* f \in \Omega^0((PU, \text{pl}_{PU}))$$

is thin invariant in the sense of thin invariance for 0-forms on path spaces over the manifold U .

1.4.2 Iterated Integrals

[...]

Definition 10 iterated integral

$$\int_{\gamma} (\omega_1, \dots, \omega_n) = \dots$$

Definition 11 Parallel transport is the map

$$\begin{aligned} \mathcal{P} \exp(f) : \Omega^1((X, \text{pl}_X), \text{Lie}(G)) &\longrightarrow \Omega^0((PX, \text{pl}_{PX}), G) \\ A &\longmapsto \mathcal{P} \exp(\int A) \end{aligned}$$

that sends a 1-form A on X to a 0-form $\mathcal{P} \exp(\int A)$ on PX which on every plot

$$V \xrightarrow{\psi} PX$$

is defined by

$$\begin{aligned} \psi^* \mathcal{P} \exp(\int A) : V &\longrightarrow G \\ \gamma &\longmapsto \mathcal{P} \exp\left(\int_{\{\gamma\} \times I} (\psi \times \text{Id} \circ \text{ev})^* A\right) \\ &= \int_{\psi(\gamma)} (A) + \int_{\psi(\gamma)} (A, A) + \int_{\psi(\gamma)} (A, A, A) + \dots \end{aligned}$$

[...]

1.5 Holonomy on Smooth Spaces

Recall the definition of ordinary smooth holonomy on a manifold:

Definition 12 Let (M, pl_M) be a manifold and let (PM, pl_{PM}) be the path space over M . Let

$$\text{hol} \in \Omega^0((PM, \text{pl}_{PM}), G)$$

be a smooth 0-form on PM which is thin invariant, maps PM onto the Lie group G and respects composition of paths. Such a 0-form hol is called a smooth **local holonomy** over M .

Holonomy on ordinary manifolds is well understood. We wish to generalize the following fact to arbitrary smooth spaces:

Proposition 2 (...) Every local holonomy hol on M comes from the parallel transport of some 1-form $A \in \Omega^1(M, \text{Lie}(G))$:

$$\text{hol} = \mathcal{P} \exp\left(\int A\right).$$

Conversely, every parallel transport on M gives a local holonomy on M and $\text{hol} = \text{hol}'$ is equivalent to $A = A'$.

Note: The terms *holonomy* and *parallel transport* are sometimes used slightly differently than defined here. In the present context, parallel transport is the assignment of 0-forms to 1-forms by path ordered exponentiated integration, while holonomy is the assignment of group elements to thin-homotopy classes of paths.

The next proposition asserts that the above statement generalizes to the case where the base manifold M is replaced by any smooth space (X, pl_X) . The idea is once again to show that the above statement holds on the domain U of each plot $\phi \in \text{pl}_X$.

Definition 13 *Let (X, pl_X) be a smooth space and let (PX, pl_{PX}) be the path space over X . Let*

$$\text{hol} \in \Omega^0((PX, \text{pl}_{PX}), G)$$

*be a smooth 0-form on PX which is thin invariant, maps PX onto the Lie group G and respects composition of paths. Such 0-forms hol are called smooth **local holonomy** functors over X .*

The main claim to be discussed here is now the following:

Proposition 3 *Every local holonomy hol on (X, pl_X) comes from the parallel transport of some 1-form $A \in \Omega^1((X, \text{pl}_X), \text{Lie}(G))$:*

$$\text{hol} = \mathcal{P} \exp \left(\int A \right).$$

Conversely, every parallel transport on X gives a local holonomy on X and $\text{hol} = \text{hol}'$ is equivalent to $A = A'$.

Proof.

First note the following

Lemma 1 *Let (PX, pl_{PX}) be the path space of (X, pl_X) . Every plot $V \xrightarrow{\phi} PX \in \text{pl}_{PX}$ factors through a smooth map $PU \xrightarrow{Pf} PX$ for $U \xrightarrow{f} X \in \text{pl}_X$ a plot of X . More precisely, given ϕ and f there is a plot $\alpha_\phi \in \text{pl}_{P(V)}$ such that*

$$\begin{array}{ccc} V & \xrightarrow{\alpha_\phi} & PU \\ \phi \downarrow & & \downarrow Pf \\ PX & = & PX \end{array} .$$

Proof.

Since $V \xrightarrow{\phi} PX$ is a plot of PX we have by definition of pl_{PX} that $V \times I \xrightarrow{\phi \times \text{Id}} PX \times I \xrightarrow{\text{ev}} X$ is a plot of X . Set $U = V \times I$ and $f = \phi \times \text{Id} \circ \text{ev}$.

Now, (PU, pl_{PU}) has a special plot

$$V \xrightarrow{\alpha_\phi} P(V \times I)$$

with domain V itself that is given by

$$\alpha_\phi(\gamma) = (\tau \mapsto (\gamma, \tau)) .$$

To check that this is indeed a plot we need to check, according to the definition of plots on path spaces, that

$$V \times I \xrightarrow{\alpha_\phi \times \text{Id}} P(V \times I) \times I \xrightarrow{\text{ev}_{P(V \times I)}} V \times I$$

is a plot of $V \times I$. But since

$$(\gamma, \sigma) \xrightarrow{\alpha_\phi \times \text{Id}} ((\tau \mapsto (\gamma, \tau)), \sigma) \xrightarrow{\text{ev}_{P(V \times I)}} (\gamma, \sigma)$$

is just the identity on $V \times I$, this is trivially true.

Now note that $Pf = P(\phi \times \text{Id} \circ \text{ev})$ acts as

$$\begin{aligned} Pf : P(V \times I) &\longrightarrow PX \\ \gamma &\longmapsto (\sigma \mapsto \text{ev}((\phi \times \text{Id})(\gamma(\sigma)))) . \end{aligned}$$

This is easily seen to imply the commutativity of the diagram stated in the lemma:

$$\begin{aligned} \gamma \xrightarrow{\alpha_\phi} \underbrace{(\tau \mapsto (\gamma, \tau))}_{=\tilde{\gamma}} \xrightarrow{Pf} & (\sigma \mapsto \text{ev}((\phi \times \text{Id})(\tilde{\gamma}(\sigma)))) \\ &= (\sigma \mapsto \text{ev}((\phi \times \text{Id})(\gamma, \sigma))) \\ &= (\sigma \mapsto \text{ev}(\phi(\gamma), \sigma)) \\ &= (\sigma \mapsto \phi(\gamma)(\sigma)) \\ &= \phi(\gamma) . \end{aligned}$$

□

Now we state the proof of prop. 3:

- ‘ \Rightarrow ’: Let $\text{hol} \in \Omega^0((PX, \text{pl}_{PX}), G)$ be a local holonomy functor on X . Given any plot $U \xrightarrow{\phi} X \in \text{pl}_X$ apply the path space functor to get a smooth map

$$PU \xrightarrow{P\phi} PX$$

with which hol can be pulled back to

$$(P\phi)^*\text{hol} \in \Omega^0((PU, \text{pl}_{PU}), G) .$$

Since U is a manifold the ordinary statement of holonomy applies and it follows that hol assigns to each U a 1-form

$$A_U \in \Omega^1(U, \text{Lie}(G)) .$$

In order to show that this collection of 1-forms on plot domains constitutes a 1-form on the smooth space X , we need to check that these 1-forms are correctly related by pullbacks under morphism between plots.

Hence consider two plots $U \xrightarrow{\phi} X$, $U' \xrightarrow{\phi'} X$ of X which are related by $U \xrightarrow{f} U'$. Furthermore consider the path spaces PU and PU'

over the domains of these plots as well as a plot $V \xrightarrow{\psi} PU$ of PU . This gives rise to the following commuting diagrams:

$$\begin{array}{ccc} V & & \\ \psi \downarrow & & \\ PU & \xrightarrow{Pf} & PU' \\ P\phi \downarrow & & \downarrow P\phi' \\ PX & = & PX \end{array}$$

and

$$\begin{array}{ccc} V \times I & & \\ \psi \times \text{Id} \downarrow & & \\ PU \times I & \xrightarrow{Pf \times \text{Id}} & PU' \times I \\ \text{ev} \downarrow & & \downarrow \text{ev}' \\ U & \xrightarrow{f} & U' \\ \phi \downarrow & & \downarrow \phi' \\ X & = & X \end{array}$$

We have

$$(P\phi)^*\text{hol} = (Pf)^*((P\phi')^*\text{hol})$$

and need to show that this implies that

$$A_U = f^*A_{U'}$$

By proposition 2 we know that

$$\psi^*((P\phi)^*\text{hol}) : \gamma \mapsto \mathcal{P} \exp \left(\int_{\{\gamma\} \times I} (\psi \times \text{Id} \circ \text{ev})^* A_U \right)$$

and that

$$\psi^*((Pf)^*((P\phi')^*\text{hol})) : \gamma \mapsto \mathcal{P} \exp \left(\int_{\{\gamma\} \times I} (\psi \times \text{Id} \circ Pf \times \text{Id} \circ \text{ev}')^* A_{U'} \right).$$

By definition of Pf we have $Pf \times \text{Id} \circ \text{ev}' = \text{ev} \circ f$ and hence the last line gives

$$\dots = \mathcal{P} \exp \left(\int_{\{\gamma\} \times I} (\psi \times \text{Id} \circ \text{ev})^* f^* A_{U'} \right).$$

Since this has to hold for all ψ it follows by prop. 2 that

$$A_U = f^*A_{U'}$$

as desired.

This means that there is a 1-form

$$A \in \Omega^1((X, \text{pl}_X), \text{Lie}(G))$$

such that

$$A_U = \phi^* A$$

for all U and all plots $U \xrightarrow{\phi} X$. Therefore the above says that

$$\psi^*((P\phi)^*\text{hol}) : \gamma \mapsto \mathcal{P}\exp\left(\int_{\{\gamma\} \times I} (\psi \times \text{Id} \circ \text{ev} \circ \phi)^* A\right).$$

Using again the definition of $P\phi$ which says that $\text{ev} \circ \phi = P\phi \times \text{Id} \circ \text{ev}$, this can be rewritten as

$$\dots = \mathcal{P}\exp\left(\int_{\{\gamma\} \times I} ((\psi \circ (P\phi)) \times \text{Id} \circ \text{ev})^* A\right).$$

Since, according to lemma 1, every plot of PX is of the form $\psi \circ P\phi$ this says nothing but that hol is indeed the parallel transport of the Chen-form A :

$$\text{hol} = \mathcal{P}\exp\left(\int A\right).$$

• ‘ \Leftarrow ’

Let

$$V \xrightarrow{\phi} PX$$

and

$$V' \xrightarrow{\phi'} PX$$

be two plots of PX related by

$$\phi \xrightarrow{\lambda} \phi'.$$

Given a 1-form

$$A \in \Omega^1((X, \text{pl}_X), \text{Lie}(G))$$

we need to construct thin-invariant 0-forms hol_V and $\text{hol}_{V'}$ on V and V' , show that they are related by

$$\text{hol}_V = \lambda^* \text{hol}_{V'}$$

and that their construction is the inverse of the above construction of a 1-form from a local holonomy 0-form.

Using the plot α_ϕ from lemma 1 we get a 0-form on V from a 1-form on X by the following combination of pullbacks and parallel transport on ordinary manifolds:

$$\begin{array}{ccccc} \Omega^0(V) & \xleftarrow{\alpha_V^*} & \Omega^0(P(V \times I)) & & \\ & & \uparrow \mathcal{P}\exp(f) & & \\ & & \Omega^1(V \times I) & \xleftarrow{(\phi \times \text{Id})^*} & \Omega^1(PX \times I) \xleftarrow{\text{ev}^*} \Omega^1(X) \end{array}$$

This is the 0-form we shall call hol_V . By the definition of $\mathcal{P}\exp(f)$ it is given by

$$\begin{aligned} \text{hol}_V : V &\longrightarrow G \\ \gamma &\longmapsto \alpha_V^* \mathcal{P}\exp\left(\int (\phi \times \text{Id} \circ \text{ev})^* A\right)(\gamma) \\ &= \mathcal{P}\exp\left(\int_{\{\gamma\} \times I} (\alpha_V \times \text{Id} \circ \text{ev}_{P(V \times I)})^* (\phi \times \text{Id} \circ \text{ev})^* A\right). \end{aligned}$$

But since $\alpha_V \times \text{Id} \circ \text{ev}_{P(V \times I)} = \text{Id}_{V \times I}$ this is nothing but

$$\dots = \mathcal{P}\exp\left(\int_{\{\gamma\} \times I} (\phi \times \text{Id} \circ \text{ev})^* A\right).$$

This manifestly implies that

$$\text{hol}_V = \lambda^* \text{hol}_{V'}$$

and that the construction of hol from a 1-form A is inverse to the construction of A from hol .

□

2 Synthetic Differential Geometry

It remains to be shown that the 1-form on PM obtained from holonomy on PM comes from a 2-form on M .

It seems useful for this purpose to adopt a concept of differential forms which lends itself to an arrow-theoretic treatment. An approach to differential geometry with this property is **synthetic differential geometry** developed by Lawvere and Kock [3] and immensely generalized (to the scheme theoretic context) by Breen and Messing [8].

§2.1 is a review and discussion of some elementary concepts and results of synthetic differential geometry in its relatively lowbrow incarnation (as compared to what Breen and Messing do) as presented in work by Kock such as [4].

§2.2 contains some first steps in an attempt to translate [8] into non-scheme language.

In §2.3 a sketch is given of what should be a proof of the fact that any (smooth, strict) p -holonomy p -functor $\text{hol} : \mathcal{P}_2(M) \rightarrow G_p$ gives rise to Lie-algebra valued 1-, 2-, ... p -forms on M .

2.1 Synthetic Differential Geometry in Terms of Infinitesimals

Let \mathbb{R} be the real numbers regarded as a ring.

Let D be some set. The ring freely generated by $\mathbb{R} \cup D$ subject to the relation $d^2 = 0 \forall d \in D$ is the ring \mathbf{R} used in synthetic differential geometry if the following **Kock-Lawvere axiom** holds (see page 3 of [6]):

Axiom 1 Every ‘smooth’¹ function $f : D \rightarrow \mathbf{R}$ extends uniquely to an affine function $\mathbf{R} \rightarrow \mathbf{R}$. This means that there are unique elements $f(0), f'(0) \in \mathbf{R}$ such that

$$f : d \mapsto f(0) + d f'(0) .$$

Similarly, $D(n) \subset \mathbf{R}^n$ denotes the subset

$$D(n) = \{(d_1, \dots, d_n) \mid \text{the product of any } k+1 \text{ of the } d_i \text{ is } 0\}$$

called the **1-monad** around $0 \in \mathbf{R}^n$.

Two elements $\vec{x}, \vec{y} \in \mathbf{R}^n$ are called **infinitesimal neighbours** if their difference lies in $D(n)$. One writes

$$\vec{x} \sim \vec{y} \Leftrightarrow \vec{x} - \vec{y} \in D(n) .$$

One important difference between synthetic differential calculus and similar approaches (like nonstandard analysis) is that this neighbourhood relation is *not transitive*. In the language introduced here this is due to the fact that elements in D are nilpotent but do *not anticommute*. Hence

¹There must be some fine print here which I need to clarify. I must admit that I still haven’t been able to look at Kock’s book [3] but are relying on scattered information from articles available online.

Example 2 For $n = 1$ let $d_1, d_2 \in D$ with $d_1 \neq d_2$. Then $d_1 - 0 = d_1 \in D$ and $d_2 - 0 = d_2 \in D$ so that $d_1 \sim 0$ and $d_2 \sim 0$. But $d_1 - d_2 \notin D$ since $(d_1 - d_2)^2 = -2d_1d_2$. Hence $d_1 \not\sim d_2$.

The main point of these definitions for the purpose of defining differential forms is that they allow to talk about **infinitesimal simplices**.

Definition 14

1. A **k -simplex** in \mathbf{R}^n is an ordered $k+1$ -tuple $(\vec{x}_0, \dots, \vec{x}_k) \in (\mathbf{R}^n)^{k+1}$ of vertices.
2. A **degenerate k -simplex** in \mathbf{R}^n is a k -simplex $(\vec{x}_0, \dots, \vec{x}_k)$ for which at least two elements coincide, $\exists i, j : \vec{x}_i = \vec{x}_j$.
3. An **infinitesimal k -simplex** in \mathbf{R}^n is a k -simplex such that all vertices are pairwise first order infinitesimal neighbours, $\forall i, j : \vec{x}_i \sim \vec{x}_j$.
4. An **infinitesimally spanned k -simplex** in \mathbf{R}^n is a k -simplex such that all vertices are first order infinitesimal neighbours of the base vertex: $\forall i : \vec{x}_i \sim \vec{x}_0$.

A k -simplex $(\vec{x}_0, \dots, \vec{x}_k)$ is said to be **based** at the vertex \vec{x}_0 . The set of **infinitesimal k -simplices based at 0** is denoted

$$\Lambda^k D(n) = \left\{ (\vec{x}_1, \dots, \vec{x}_k) \in \mathbf{R}^k \mid (0, \vec{x}_1, \dots, \vec{x}_k) \text{ is an infinitesimal } k\text{-simplex} \right\}.$$

Similarly we have the **infinitesimally spanned k -simplices based at 0**:

$$\tilde{\Lambda}^k D(n) = \left\{ (\vec{x}_1, \dots, \vec{x}_k) \in \mathbf{R}^k \mid (0, \vec{x}_1, \dots, \vec{x}_k) \text{ is an infinitesimally spanned } k\text{-simplex} \right\}.$$

The main point of all this is that it allows to talk about p -forms in terms of functions on p -simplices which are *not necessarily linear* in their arguments. (Compare the second paragraph in the introction to [8]) This allows to work in terms on not necessarily linear functions on arbitrary simplices that *represent* a given map on infinitesimal simplices, which again defines a p -form.

Proposition 4 ([3], recalled in [4]) Any map $\omega : \Lambda^k D(n) \rightarrow \mathbf{R}$ on infinitesimal simplices based at 0 for which

$$\exists i : \vec{x}_i = 0 \Rightarrow \omega(\vec{x}_1, \dots, \vec{x}_k) = 0,$$

is the restriction of a unique k -linear alternating map $\bar{\omega} : (\mathbf{R}^n)^k \rightarrow \mathbf{R}$.

This statement was later extended as follows

Proposition 5 ([5]) The following sets of maps are in bijection:

1. multilinear alternating maps on p -simplices (to \mathbf{R})
2. alternating maps from infinitesimally spanned simplices that vanish on degenerate simplices
3. maps on infinitesimal simplices that vanish on degenerate simplices.

Note that the truth of these statement depends crucially on two curious properties of the above formalism

- The fact that infinitesimals do *not* anticommute.
- The fact that an infinitesimal simplex is defined to be a simplex in which *any* two vertices are infinitesimal neighbours.

Corollary 1 *Any function on infinitesimal simplices which vanishes when the base vertex appears twice actually vanishes on all degenerate infinitesimal simplices.*

Hence one makes the following

Definition 15 *A p -form with values in G is a map ω from infinitesimal p -simplices to a Lie group G such that it maps degenerate simplices to the unit element of G .*

Note that the Lie algebra of G does not make any explicit appearance in this definition.

p -forms as conceived in this language are often addressed as **combinatorial p -forms**.

Proposition 6 ([4]) *Combinatorial p -forms with values in G are alternating in the sense that*

$$\omega(\dots, \vec{x}_i, \dots, \vec{x}_j, \dots) = (\omega(\dots, \vec{x}_j, \dots, \vec{x}_i, \dots))^{-1} .$$

(The inverse on the right is that in G .)

Example 3

A combinatorial G -valued 1-form ∇

$$(x, y) \mapsto \nabla(x, y)$$

may be addressed as a local connection 1-form.

Associated with it is a 2-form, namely its **curvature 2-form** $\exp(F_\nabla)$ which in the synthetic formalism has the nice expression

$$\exp(F_\nabla)(x, y, z) = \nabla(x, y) \nabla(y, z) \nabla(z, x) ,$$

(all products are simply products in G). This is nothing but the monodromy of ∇ around an (infinitesimal) 2-simplex (see section 9 of [4]).

It is instructive to check that this satisfies prop. 6. By pure combinatorics (hence the name of these p -forms) one finds

$$\exp(F_\nabla)(x, z, y) = \exp(F_\nabla(x, y, z))^{-1}$$

even on macroscopic simplices, as well as

$$\begin{aligned} \exp(F_\nabla)(y, x, z) &= \nabla(y, x) (F_\nabla(x, y, z))^{-1} \nabla(x, y) \\ &= \exp(F_\nabla(x, y, z))^{-1} . \end{aligned}$$

Here the last equality holds (only) on infinitesimal simplices, since the conjugation by $\nabla(x, y)$ introduces higher orders of the infinitesimal $\vec{x} - \vec{y} \in D(n)$.

The concept of having an arbitrary function on simplices and then restricting it to *infinitesimal* simplices in the synthetic sense of course corresponds to nothing but passing to the differentials of these functions. The point is that the above tells us essentially that the differentials of functions on simplices are alternating in their arguments already if these functions vanish on degenerate simplices.

Now one can set up all the usual machinery on differential forms in this convenient context:

[...]

2.1.1 Exponentials and Logarithms

In order to relate the combinatorial group-valued p -forms defined above to the familiar Lie-algebra-valued p -forms one has to compose them with the logarithm on \mathbf{R} . See section 2 of [4].

2.2 Combinatorial Differential Forms for Mortals

This subsection is supposed to recall the more highbrow setup of synthetic differential geometry due to [8].

This subsection is highly unfinished. If at all, read at your own risk!

Let M be a manifold and G a Lie group.

Let

$$C^\infty(M) := C^\infty(M, G)$$

be the ring of smooth functions from M to G .

There is a ring homomorphism

$$m : \begin{array}{ccc} C^\infty(M \times M) & \rightarrow & C^\infty(M) \\ ((x, y) \mapsto f(x, y)) & \mapsto & (x \mapsto f(x, x)) \end{array} . \quad (1)$$

Let J be the ideal

$$J = \ker(m)$$

of functions in $C^\infty(M \times M)$ that evaluate to the unit $1 \in G$ on the diagonal $\{(x, x) \in M \times M | x \in M\}$.

With the example where G is the additive group of real numbers in mind, we also say that such functions ‘*vanish*’ on the diagonal. Similarly, we will allow ourselves to write $f - g$ for fg^{-1} .

$J^n \subset C^\infty(M)$ is the ideal of functions which vanish up to their $(n-1)$ st derivative on the diagonal.

There is an exact sequence of rings

$$0 \rightarrow J \rightarrow C^\infty(M \times M) \xrightarrow{m} C^\infty(M) \rightarrow 0 .$$

The ideal J is generated by functions

$$1 \otimes j - j \otimes 1 : \begin{array}{ccc} M \times M & \rightarrow & G \\ (x, y) & \mapsto & j(y) - j(x) \end{array}$$

for $j \in C^\infty(M)$. A general element of J is hence of the form

$$\sum_i a_i(1 \otimes j_i - j_i \otimes 1) := \sum_i (a_i \otimes j_i - a_i j_i \otimes 1)$$

with $a_i, j_i \in C^\infty(M)$.

The ring

$$C^\infty(M \times M)/J^2$$

is the ring of equivalence classes of smooth functions on $M \times M$ that agree on the diagonal up to their first derivative. This is the ring of **first order jets** on M .

Let $f \in C^\infty(M \times M)$ be any representative of an element of this ring and assume we have identified M with \mathbb{R}^n by a choice of global coordinates, then we get uniquely (for this choice of coordinates) another representative

$$\tilde{f} := (x, y) \mapsto f(x, x) + \mathbf{d}_y f(x, x)(\vec{y} - \vec{x}) .$$

The spectrum of this ring (the set of ring homomorphisms to the group algebra of G) is called the **first infinitesimal neighborhood** of M

$$\Delta^1 := \text{Spec}(C^\infty(M \times M)/J^2) .$$

There is another exact sequence of rings

$$0 \rightarrow J/J^2 \rightarrow C^\infty(M \times M)/J^2 \xrightarrow{m} C^\infty(M) \rightarrow 0 .$$

The ring

$$\Omega^1(M) := J/J^2$$

is nothing but the ring of 1-forms on M , an observation that goes back to Kähler.

This can be seen as follows:

The representatives of equivalence classes in J/J^2 all vanish on the diagonal and are characterized purely by their first Taylor coefficient evaluated on the diagonal which is given by a 1-form on M : We can write (assuming we have locally identified M with \mathbb{R}^n)

$$\begin{aligned} & \sum_i (a_i \otimes j_i - a_i j_i \otimes 1)(x, y) \\ = & \sum_i (a_i \otimes j_i - a_i j_i \otimes 1)(x, x) + \sum_i (a_i \otimes (\mathbf{d}j_i)(y - x))(x, x) + \dots \end{aligned}$$

Hence $\sum_i (a_i \otimes j_i - a_i j_i \otimes 1)(x, y)$ corresponds to the 1-form $\sum_i a_i \mathbf{d}j_i$.

In addition to the ring homomorphism (1) we have the two homomorphisms

$$\begin{array}{ccc} C^\infty(M) & \rightarrow & C^\infty(M \times M) \\ b & \mapsto & b \otimes 1 \end{array}$$

and

$$\begin{array}{ccc} C^\infty(M) & \rightarrow & C^\infty(M \times M) \\ b & \mapsto & 1 \otimes b \end{array} .$$

Any ring homomorphism

$$R \xrightarrow{\phi} S$$

gives rise to a map

$$\text{Spec}(S) \xrightarrow{\phi_*} \text{Spec}(R) .$$

For let $S \xrightarrow{s} \mathbb{R} \in \text{Spec}(S)$ then $R \xrightarrow{\phi \circ s} \mathbb{R} \in \text{Spec}(R)$

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \downarrow \phi \circ s & & \downarrow s \\ \mathbb{R} & \xlongequal{\quad} & \mathbb{R} \end{array} .$$

We can divide out by J^2 in all the three ring homomorphism which are available. With $\text{Spec}(C^\infty(M)) = X$ and $\text{Spec}(C^\infty(M \times M)/J^2) = \Delta^1$ (by definition) we hence get maps

$$p_0, p_1 : \Delta^1 \rightarrow X$$

and

$$\Delta : X \rightarrow \Delta^1 .$$

[...]

2.3 p -Forms from p -Holonomy Functors

Here I try to demonstrate (this is in progress and needs to be extended, tuned and scrutinized) how synthetic differential geometry allows to elegantly extract from any smooth thin-invariant p -holonomy p -functor the differential 1-, 2-, ..., p -forms it comes from.

The idea is the following. The properties of a smooth and thin invariant p -functor are precisely those that allow us to construct smooth group-valued functions on p -simplices that vanish on degenerate simplices. Extending these functions from \mathbb{R} to \mathbf{R} and then restricting them to infinitesimal p -simplices yields, by def. 15 (p. 15), a p -form with values in the corresponding group.

It suffices for our purposes to work locally. This means that we can and will identify target space $M \simeq \mathbb{R}^n$ throughout, without loss of generality.

Let $P^p M$ be the space of p -paths in M . Given any p -simplex Δ in M we can associate an element in $P^p M$ to it that fills it in the sense of standard p -simplices in \mathbb{R}^n . In fact, we can do this in a way that varies smoothly with the choice of p -simplex. In other words, given a *choice of parameterization*, the space $M^{p+1} = (\mathbb{R}^n)^{p+1}$ of p -simplices equipped with the choice of parameterization is a *plot* for $P^p M$.

This again means that evaluating a local p -holonomy p -functor on these p -paths yields a smooth Lie group valued map (in the ordinary sense) on the space of p -simplices in M . Such a map is a candidate combinatorial p -form! This is making crucial use of the fact that p -holonomy is a smooth map in the sense of Chen.

The other crucial property of p -holonomy is thin invariance. This property is equivalent to the pair of properties

1. invariance under orientation-preserving reparameterization
2. inversion under orientation-reversing reparameterizations.

The first of these ensures that the combinatorial p -form to be obtained from the p -holonomy p -functor is independent of the choice of parameterization that we made above and is hence well defined.

The second of these properties finally implies that this function maps degenerate simplices to the unit group element. According to def. 15 this means that evaluating the p -holonomy p -functor on p -simplices in the above sense indeed yields a combinatorial p -form.

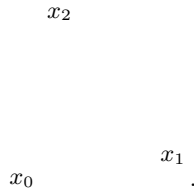
(Unless I am confused, that is. :-)

The following subsections will provide more details on this procedure.

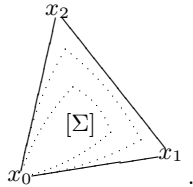
2.3.1 Simplices as p -Morphisms

We need to specify a way how to get a p -morphism in $\mathcal{P}_p(M)$ from a p -simplex in M .

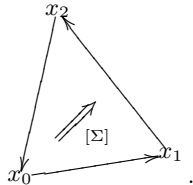
A p -simplex (assume $p = 2$ for illustration purposes) $(x_0, x_1, x_2) \in M^{p+1}$ is a tuple of points



When filled by a p -path Σ as described above, it looks like

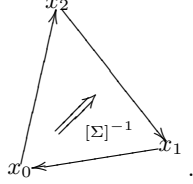


It is based at x_0 and the choice of order of its further arguments determines a choice of orientation which we shall denote this way:

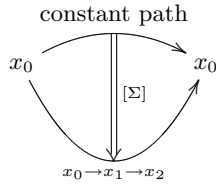


Exchanging the last two elements of the tuple yields the simplex (x_0, x_2, x_1)

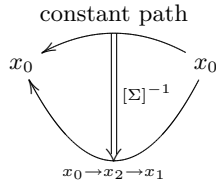
which is still based at x_0 but oriented oppositely:



Hence we want to interpret this p -simplex as a p -morphism



for one orientation, or as a 2-morphism



for the other. Here $[\Sigma]^{-1}$ denotes the inverse under horizontal composition in $\mathcal{P}_2(M)$. The horizontal composite $[\Sigma] \cdot [\Sigma]^{-1}$ is thin-homotopic to the constant 2-path on x_0 .

2.3.2 1-Connections

Pretty obvious, will write that out later.

2.3.3 2-Connections

Definition 16 Given a local 2-holonomy 2-functor $\text{hol} : \mathcal{P}_2(M) \rightarrow G_2 = (H \xrightarrow{t} G)$ we get maps

$$\nabla_{\text{hol}} : M^2 \rightarrow G$$

and

$$\nabla_{\text{hol}} : M^3 \rightarrow H$$

(which we allow ourselves to distinguish only by their domain of arguments) by

$$\nabla_{\text{hol}}(x, y) := \text{hol} \left(\begin{array}{c} x \longrightarrow y \end{array} \right)$$

$$\nabla_{\text{hol}}(x, y, z) := \text{hol} \left(\begin{array}{c} \begin{array}{ccc} & z & \\ x & \nearrow & y \\ & \searrow & \\ & & \end{array} \\ \begin{array}{c} \text{[}\Sigma\text{]} \end{array} \end{array} \right)$$

Proposition 7 *The map ∇_{hol} represents a combinatorial 1-form*

$$\nabla(x, y) := \exp(A(x, y))$$

and a combinatorial 2-form

$$\nabla(x, y, z) := \exp(B(x, y, z))$$

which satisfy what is called the fake-flatness condition (see [9])

$$t(\exp(B(x, y, z))) = \exp(F_A(x, y, z)) .$$

Proof.

By arguments of the kind given above it follows that we indeed have a combinatorial 1-form and 2-form. The fake-flatness constraint follows in the synthetic formalism easily from

$$\begin{aligned} t(\text{hol}(\Sigma)) &= \text{hol}(\gamma_{xyz}) \\ \Rightarrow t(\exp(B(x, y, z))) &= \exp(F_A(x, y, z)) . \end{aligned}$$

□

So by restricting ∇_{hol} to infinitesimal simplices we have obtained a map of type 3 in prop. 5. But more is true. It suffices to restrict ∇_{hol} to infinitesimally spanned simplices, since this yields a map of type 2 in that proposition. Here are some more details:

Note that ∇_{hol} reverses sign under an odd number of pairwise permutations of its arguments.

Let

$$\text{hol} \left(\begin{array}{c} z \\ \nearrow \quad \searrow \\ x \quad \quad y \\ \text{[}\Sigma\text{]} \end{array} \right) = \begin{array}{ccc} & \text{Id} & \\ E_x & \begin{array}{c} \curvearrowright \\ \downarrow h \\ \curvearrowleft \end{array} & E_x \\ & \text{hol}(\gamma_{xyz}) & \end{array} .$$

Permuting the last two arguments corresponds to taking the horizontal inverse

$$\text{hol} \left(\begin{array}{c} z \\ \nearrow \quad \searrow \\ x \quad \quad y \\ \text{[}\Sigma\text{]} \end{array} \right) = \begin{array}{ccc} & \text{Id} & \\ E_x & \begin{array}{c} \curvearrowright \\ \downarrow h^{-1} \\ \curvearrowleft \end{array} & E_x \\ & \text{hol}(\gamma_{xzy}) & \end{array} .$$

Cyclically permuting all three arguments corresponds to conjugating with $x \xrightarrow{\text{hol}(\gamma_{xy})} y$

$$\begin{array}{c} z \\ \nearrow \quad \searrow \\ x \quad \quad y \\ \text{[}\Sigma\text{]} \end{array} \xrightarrow{\text{hol}(\gamma_{xy})} \begin{array}{c} z \\ \nearrow \quad \searrow \\ x \quad \quad y \\ \text{[}\Sigma\text{]} \end{array}$$

Hence

$$\begin{aligned}
 \text{hol} \left(\begin{array}{c} z \\ \swarrow \quad \searrow \\ x \quad y \\ \nearrow \quad \nwarrow \\ x \quad y \end{array} \right) &= \text{hol} \left(\begin{array}{c} z \\ \swarrow \quad \searrow \\ x \quad y \\ \nearrow \quad \nwarrow \\ x \quad y \end{array} \right) \\
 &= E_y \xrightarrow{\text{hol}(\gamma_{yx})} E_x \begin{array}{c} \xrightarrow{\text{Id}} \\ \downarrow h \\ \xrightarrow{\text{hol}(\gamma_{xy})} \end{array} E_x \xrightarrow{\text{hol}(\gamma_{xy})} E_y \\
 &= E_y \begin{array}{c} \xrightarrow{\text{Id}} \\ \downarrow h' \\ \xrightarrow{\text{hol}(\gamma_{yx})} \end{array} E_y,
 \end{aligned}$$

where

$$h' = \alpha(\text{hol}(\gamma_{yx}))(h).$$

This represents the same 2-form as h . On infinitesimal 2-simplices the action $\alpha(\text{hol}(\gamma_{yx}))$ drops out (compare example 3 (p. 15)).
[...]

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