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Loop Space Methods in String Theory

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Abstract: The constraints of the type II string can be regarded as generalized deRham and Laplace-Beltrami operators on loop space. From this perspective nontrivial target space background fields are related to Morse-theory-like deformations of loop space differential geometry.

Some applications of this approach are discussed, concerning computation of superstring spectra in non-exactly solvable backgrounds, duality transformations on target space fields, construction of boundary states for branes with fields turned on and the relation to Pohlmeyer and DDF invariants, as well as strings in nonabelian 2-form backgrounds and the corresponding nonabelian 1-gerbe connections.

It is furthermore shown how the deformations of the superconformal generators can be equivalently regarded as deformations of the Hodge star operator on loop space, and hence of the inner product over loop space differential forms. This allows to associate a "weak" loop space spectral triple with every consistent superstring background.

As a toy example of generalized backgrounds in this sense a discrete target space is studied in the superparticle limit. The loop space deformation technique is shown to admit a mimetic NCG analogue of the Hodge star operator and to single out a target space geometry with Lorentzian signature, similar to causal sets.

Contents

LAB: -TOC-

This is a review of work that has been reported in [1, 2, 3, 4, 5, 6, 7, 8] and is based on talks given by the author in July 2004 at the conference *Non-commutative geometry and representation theory in mathematical physics* in Karlstad university, at the theory seminar of University of Bonn and at University Duisburg-Essen.

CIT: Schreiber:2004,Schreiber:2

This is an unfinished draft, see `http://www-stud.uni-essen.de/~sb0264/looptalk.pdf`.

1. Introduction

String theory has taught that there is an intimate relationship between extended relativistic objects, supersymmetry, and generalized geometry. Much of this connection manifests itself in the encoding of spacetime backgrounds in terms of two-dimensional superconformal field theories. These are, not surprisingly, commonly studied in field theoretic language and in particular in the Heisenberg picture of quantum theory. When one instead switches to the equivalent Schrödinger picture string evolution looks like the propagation of a relativistic *point* in loop space, the space of maps from a worldsheet *slice* into spacetime. This (canonical) formulation has received much less attention than genuine CFT techniques. The purpose of this paper is to review some recent constructions and results that are naturally derived and formulated in loop space.

A pivotal role in this endeavor is played by the demonstration that the relationship between consistent worldsheet theories and target space backgrounds manifests itself in terms deformations of the Hodge inner product on the exterior bundle over loop space, induced by the target space fields. This gives an algebraic handle on strings in background fields which facilitates a number of constructions that yield insight into string dynamics. An overview is given in the following subsections.

Relation to existing literature. The loop space perspective on superstrings presented here, in combination with the geometrical deformation technique and its implications, is original, but of course several aspects of this approach have been discussed before:

Some crucial remarks on the formulation of superstring theory in loop space have been made in the respective second halfs of [9, 10]. Mostly in the context of studies of the Dirac-Ramond operator, loop space has been discussed in more detail for instance in [11, 12]. Aspects of the theory of differential forms on loop space are studied in [13].

CIT: Witten:1982,Witten:1985■

CIT: Wurzbacher:1995,Wurzba

CIT: GetzlerJonesPetrack:1991

The deformations that we apply on loop space operators are a direct generalization from 1+0 to 1+1 dimensions of those considered in [9]. A brief mention of a torsion deformation in 1+0 dimensions which we show below to be related to strings in Kalb-Ramond fields when lifted to 1+1 dimensions is given in [14, 15].

CIT: Witten:1982

CIT: FroehlichGrandjeanRecknagel:1996,FroehlichGran

These latter references, with their strong emphasis on the relation between supersymmetry and geometry, have also had a large motivational impact on the general framework to be discussed here. Some aspects of a relation between superstring constraints and the general geometric content of supersymmetric quantum mechanics, hinted at in the last sections of these references, reappear in the approach to be presented here, although in a slightly different fashion.

The fact that the worldsheet supercharges define two Dirac operators which encode target space backgrounds in the generalized sense of spectral geometry has been illuminated in [16, 17, 18]. In particular, in [19, 20] the canonical form of the superconformal constraints for Kalb-Ramond backgrounds, which we show follow in a single line from an appropriate deformation, are derived from a superfield Lagrangian, and the relation between the spectral action of the resulting worldsheet supercharge with the string background action is demonstrated.

CIT: Lizzi:1999,LizziSzabo:199
CIT: Chamseddine:1997,Chamseddine:1

The study of target space gauge and duality transformations by means of inner algebra automorphisms, which appear as a special case of the deformations to be discussed here, has been given in [21, 22, 16, 17, 23].

CIT: EvansGiannakis:1994,EvansGiannaki

Another construction which turns out to be a special case of the general deformations considered here is that of boundary states for branes with gauge fields turned on, given in [24, 25, 26]. We show how this relates to the notion of Pohlmeyer invariants, which were studied in [27, 28, 29, 30].

CIT: Hashimoto:2000,Hashimo
CIT: Pohlmeyer:2002,Meusbur

The nonabelian 1-gerbe connection on loop space which we derive from string dynamics in nonabelian 2-form backgrounds is similar, but not identitcal to that proposed in [31].

CIT: Hofman:2002

The "relaxed" spectral triples that we use, in order to relate the loop space deformations to deformed inner product spaces with Dirac operators, are, as far as they differ from standard spectral triples, an extension of the work on abstract differential calculi in [32, 33, 34, 35, 36] as well as inspired by the work on Krein space Lorentzian-signature spectral triples in [37].

CIT: DimakisMueller-Hoissen:1994,DimakisMue
Hoissen:1997,DimakisMue
Hoissen:1998,DimakisMue
Hoissen:2002,DimakisMue
Hoissen:2002a
CIT: Strohmaier:2001

1.1 Overview

The next subsections briefly summarize some constructions and results that have been obtained in [1, 2, 3, 4, 5, 6, 7, 8] using loop space or contexts closely related to the topics discussed here. A more detailed discussion is given in §2 (p.12) and §3 (p.20)

CIT: Schreiber:2004,Schreiber:

1.1.1 Super-Virasoro and deformations in loop space

LAB: Super-Virasoro and deformations in loop space

Let $\mathbf{d}_K$ be a generalization of the exterior derivative on *parametrized* loop space (to be described in detail in §2.2 (p.13)) which squares to the generator $i\mathcal{L}_K$ of loop reparemetrizations. Furthermore, let $\langle\cdot|\cdot\rangle$ be the Hodge inner product on differential forms on loop space with respect to the metric induced from target space, and let $\mathbf{d}^\dagger{}_K$ be the adjoint of $\mathbf{d}_K$ with respect to this inner product. The super-Virasoro constraints of the type II string are then schematically equivalent to the *Dirac-Kähler* equations

$$(\mathbf{d}_K \pm \mathbf{d}^\dagger{}_K)\,|\psi\rangle = 0\,, \tag{1.1}$$

LAB: DK constraints

for *states* $|\psi\rangle$ which are inhomogeneous differential forms on loop space.

A homomorphism of the superconformal algebra obtained from conjugation with some operator $\exp(\mathbf{W})$

$$\mathbf{d}_K \mapsto e^{-\mathbf{W}} \circ \mathbf{d}_K \circ e^{\mathbf{W}}$$

$$\mathbf{d}^{\dagger}{}_K \mapsto e^{\mathbf{W}^{\dagger}} \circ \mathbf{d}^{\dagger}{}_K \circ e^{-\mathbf{W}^{\dagger}}\,, \tag{1.2}$$

LAB: extd deformation

which preserves the nilpotence of $\mathbf{d}_K$ up to reparametrizations (this is in particular the case when $\mathbf{W}$ is itself reparametrization invariant) shall be call a *Morse-theory-like deformation*, because it is a generalization of the similar deformation which relates supersymmetric quantum mechanics to Morse theory [9].

CIT: Witten:1982

As shown in [1], all massless NS background fields of the open and closed string arise from such Morse-theory-like deformations. The hermitean part of $\mathbf{W}$ is related to the vertex operator of the corresponding field, while the anti-hermitean part induces target space gauge transformations and dualities. When (1.2) is expanded to first order the theory of *canonical deformations* [38, 39, 23] of superconformal field theories is reproduced.

CIT: Schreiber:2004

CIT: Giannakis:1999,EvansGiannaki

What is it good for to know that superstring backgrounds are associated with deformations of the form (1.2)? The point is that it facilitates addressing and answering certain problems in a convenient way. Examples for this are reviewed in the following subsections.

1.1.2 Covariant loop space Hamiltonian and string spectra

As a first example, we shall discuss the problem of finding *covariant* Hamiltonian operators for the superstring: Suppose a timelike Killing vector v on target space is given and some background fields $\{\Phi\}$ are turned on, which are invariant under the flow induced by v. Can we find an operator $H_v^{(\Phi)}$ on the Hilbert space of the covariantly quantized string such that its expectation value $\langle H_v^{\Phi}\rangle$ computes the energy of the string as measured by v? This question is of importance for the computation of string spectra in non-exactly solvable background field, as for instance are needed for tests of AdS/CFT, and has so far been addressed only in light-cone gauge formalism, e.g. [40, 41, 42]. The technique presented here [7] allows to construct similar Hamiltonians in the RNS formalism without the need for fixing light-cone or even the presence of a lightlike Killing vector. (However, in the RNS formulation of course the usual problem with RR backgrounds is encountered.)

CIT: ParnachevSahakyan:2002,ParnachevSa

CIT: Schreiber:2003a

First of all note that the constraints (1.1) imply the *Dirac-Schrödinger* equation

$$i\mathcal{L}_v\,|\psi\rangle = \underbrace{\frac{i}{4}\left(\left[v\cdot\gamma_+, \mathbf{d}_K - \mathbf{d}^{\dagger}{}_K\right] - \left[v\cdot\gamma_-, \mathbf{d}_K + \mathbf{d}^{\dagger}{}_K\right]\right)}_{:=H_v}|\psi\rangle \tag{1.3}$$

LAB: Dirac Schroedinger

describing evolution of $|\psi\rangle$ along v, where, by abuse of notation, v here denotes the lift of the timelike Killing vector to loop space. Furthermore $\mathcal{L}_v = [\mathbf{d}_K, \iota_v]$ is the Lie derivative along v and $\gamma_\pm^\mu = dx^\mu \wedge \pm\iota_{\partial_\mu}$ are the two copies of the Clifford algebra on the exterior bundle.

The operator H_v is a Hamiltonian on loop space which measures the string's energy along v. The crucial point of formula (1.3) is that it generalizes to all background field configurations which are invariant under the flow induced by v and associated with deformations of the form (1.2). One just has to transform the $v\cdot\gamma_\pm$ as

$$v\cdot\gamma_\pm \mapsto v\cdot\left(e^{\mathbf{W}^\dagger}\circ dx\wedge\circ e^{-\mathbf{W}^\dagger} \pm e^{-\mathbf{W}}\circ\iota_\partial\circ e^{\mathbf{W}}\right)\,, \tag{1.4}$$

because then the deformed constraints

$$\begin{aligned} &\left(\mathbf{d}_K^{(\Phi)} \pm \mathbf{d}^{\dagger}{}_K^{(\Phi)}\right)|\psi\rangle = 0 \\ \Leftrightarrow\ &\left(e^{-\mathbf{W}}\circ\mathbf{d}_K\circ e^{\mathbf{W}} \pm e^{\mathbf{W}^\dagger}\circ\mathbf{d}^\dagger{}_K\circ e^{-\mathbf{W}^\dagger}\right)|\psi\rangle = 0 \end{aligned} \tag{1.5}$$

imply the deformed Dirac-Schrödinger equation

$$i\mathcal{L}_v\,|\psi\rangle = \underbrace{\frac{i}{4}\left(\left[v\cdot\gamma_+^{(\Phi)}, \mathbf{d}_K^{(\Phi)} - \mathbf{d}^{\dagger}{}_K^{(\Phi)}\right] - \left[v\cdot\gamma_-, \mathbf{d}_K^{(\Phi)} + \mathbf{d}^{\dagger}{}_K^{(\Phi)}\right]\right)}_{:=H_v^{(\Phi)}}|\psi\rangle\ . \tag{1.6}$$

LAB: deformed Dirac Sch

Up to some subtleties, this formalism therefore allows to do ordinary quantum mechanical perturbation theory to derive shifts in the superstring spectra as background fields are turned on.

In [7] this was demonstrated by calculating energy shifts for superstrings on $\mathrm{AdS}_3\times S^3$ away from its pp-wave Penrose limit. Application to $\mathrm{AdS}_5\times S^5$ would be very interesting but requires a better understanding of how to incorporate RR-background fields by means of (1.2) (but see [43]).

CIT: Schreiber:2003a

CIT: Giannakis:2002

1.1.3 Computation of duality relations for general background fields

It has been noted in [22], in the context of the bosonic string, that T-duality relations of arbitrary background fields are conveniently studied in canonical formalism, where it amounts to swapping canonical momenta $P_\mu(\sigma)$ with the σ-derivative of the canoncial coordinates $X'^\mu(\sigma)$. It can be shown that this transformation is in fact obtained from an inner automorphism of the operator algebra induced by a unitary deformation $\exp\left(\mathbf{W}^{(T)}\right)$ (1.2), in accordance with with general fact that hermitean deformation operators $\mathbf{W}$ relate physically equivalent backgrounds of the *closed* string (*cf.* §1.1.1 (p.3) but also note that for the *open* string this does not hold in general, as dicussed below in §1.1.4 (p.6)).

CIT: EvansGiannakis:1996

In [1] it has been shown how the *Buscher rules* for T-duality along a single direction on metric and Kalb-Ramond fields are obtained with minimal computational effort by applying the above unitary transformation to the superconformal generators in their *canonical* form as they appear in loop space formalism. Knowing the

CIT: Schreiber:2004

canonical form of the constraints from deformations (1.2) for more general backgrounds and applying the above automorphism to them hence allows a convenient way to compute the action of T-dualtiy on these background configurations. This is used in [1] to determine the physical interpretation of the background induced by the deformation which is adjoint to that which gives the Kalb-Ramond field. The results suggest that this has to be interpreted as giving the D-string coupled to the RR-form, as opposed to the F-string coupled to the NS 2-form.

CIT: Schreiber:2004

1.1.4 SCFT and boundary state deformation

LAB: SCFT and boundary state deformation

A further application of (1.2) is the construction of boundary states. First of all one notes that the bare boundary state of a flat, unexcited $D9$ brane is nothing but the constant 0-form on loop space. This suggests that loop space formalism is natural for boundary states.

Indeed, while for closed strings all unitary deformation operators $\exp(\mathbf{W})^\dagger = \exp(-\mathbf{W})$ correspond to pure gauge transformations, these transformations may describe genuinely new physics when applied to open strings. A unitary deformation (1.2) of the contraint operators can be absorbed in a corresponding transformation

$$|\psi\rangle \mapsto \exp(-\mathbf{W})\,|\psi\rangle \tag{1.7}$$

LAB: unitary deformation

of the closed string states. But in boundary state formalism amplitudes $\langle\cdots\rangle_{\text{open}}$ for open strings attached to a given D-brane are computed as closed string amplitudes $\langle\cdots|\alpha\rangle_{\text{closed}}$ with a suitable closed string boundary state source $|\alpha\rangle$ included. If this is transformed as in (1.7) to $\langle\cdots e^{-\mathbf{W}}\,|\alpha\rangle_{\text{closed}}$ a new physical configuration is described, which is not gauge trivial.

It was indeed noted in [24, 25] that the boundary state describing a brane with an abelian gauge field is obtained by acting with the the unitary Wilson loop operator $\exp\left(i\oint A\cdot X'\right) +$(fermionic terms) along the closed string on the bare boundary state. This is precisely the deformation operator $\exp(\mathbf{W})$ found in [1] to describe the gauge trivial effect of turning on a background gauge field for the closed string, so that the above general picture is confirmed.

CIT: Hashimoto:2000,Hashimot

CIT: Schreiber:2004

This demonstrates in a specific example how the deformation formalism (1.2) applies to boundary states and hence open strings. Heuristically, we see that in going from closed to open strings the inner product is “cut open” and only one side of (1.2) applies.

This phenomenon can alternatively be understood from string field theory. As was shown in [44] (in the context of bosonic OSFT) a classical string field solution which is formally pure gauge gives rise to a unitary transformation of the BRST operator (and hence of its constituents, the superconformal generators) which can, by a unitary field redefinition, be pulled off to the boundary, where it deforms the boundary state.

CIT: Kluson:2002

1.1.5 Deformations by Pohlmeyer invariants

One may ask which deformation operators in general produce sensible boundary states. Since the Ishibashi conditions demand that a boundary state be annihilated by $\mathcal{L}_K$ as well as $\mathbf{d}_K$ (R-sector) or $\mathbf{d}^\dagger{}_K$ (NS sector), it is reasonable to look for unitary operators $\exp(\mathbf{W})$ which commute with *all* of the super-Virasoro constraints, since this guarantees a map from admissible boundary states to admissible boundary states. With respect to a gauge field background this singles out what for the bosonic string is known as *Pohlmeyer invariants* [27, 28, 29, 30], which are generalized holonomies of the gauge field along the string, where the exponent contains not $A{\cdot}X'$ but $A{\cdot}(iP \pm X')$, with P the canonical worldsheet momentum.

CIT: Pohlmeyer:2002,Meusburg

The literature on Pohlmeyer invariants is mostly concerned with attempts to find an alternative quantization of string in terms of these objects. However, the set of all Pohlmeyer invariants is just a subset of the set of the ordinary DDF invariants [2, 3].

CIT: Schreiber:2004b,Schreiber

Still, they do have some interesting properties. One can check [5] that when generalized to the superstring, as discussed in [2], and when applied to the D9-brane boundary state, following the disucssion in §1.1.4 (p.6), they reproduce the boundary state for nonabelian gauge fields that was constructed and studied in [26]. As shown there, this boundary state is finite and well defined only when the background equations of motion for the gauge field hold. For the Pohlmeyer invariants this are the equations of fully dimensionally reduced SYM, which elucidates the relation of the Pohlmeyer invariants to the IKKT model noted in [2].

CIT: Schreiber:2004d
CIT: Schreiber:2004b
CIT: MaedaNakatsuOonishi:2004
CIT: Schreiber:2004b

1.1.6 Nonabelian 2-form gauge fields and 1-gerbe connections

Instead of using boundary state formalism it is possible to apply the loop space formalism and its deformation theory directly to open strings, by considering the space of (smooth) maps from the (parametrized) *interval* into target space. Everything goes through as before, except for possible boundary terms. For instance when applying the gauge field deformation $\exp\big(\mathbf{W}^{(A)}\big)$ in the open string case $\mathbf{d}_K^{(A)}$ picks up boundary terms at both ends of the string which are plus/minus the usual gauge covariant derivatives with respect to A for the endpoints, just as expected.

This suggests the more detailed study of nonabelian deformations and in particular of those including a nonabelian 2-form field. Since now the (generalized) Wilson lines are along the open interval, the resulting operator has to act on a suitable bundle over loop space.

Deformations of the closed string superconformal generators show that the grade 1 part of $\mathbf{d}_K^{(\Phi)}$ is the connection on loop space induced by the background fields $\{\Phi\}$. When this is computed for $\{\Phi\}$ a nonabelian gauge field and 2-form background one finds [6] a nonablelian 2-form connection on loop space which has the properties expected from the theory of crossed modules/2-groups [31, 45, 46, 47]. Hence this

CIT: Schreiber:2004e
CIT: Hofman:2002,Baez:2002,Lahi

way a boundary SCFT is found which should describe open strings in nonabelian 2-form backgrounds.

1.1.7 Relation to spectral/noncommutative geometry

From the differential geometric viewpoint it may seem unnatural to have an exterior derivative $\mathbf{d}_K^{(\Phi)}$ which depends on the background fields [48], since in ordinary geometry $\mathbf{d}$ is independent of the metric while all the geometric information is carried by the Hodge inner product and the adjoint $\mathbf{d}^\dagger$.

CIT: Forgy:2004

But, as discussed in [1], the deformation (1.2) is equivalent under a global similarity transformation to a deformation of the Hodge inner product of the form

CIT: Schreiber:2004

$$\langle\cdot|\cdot\rangle \mapsto \left\langle \cdot | e^{-\mathbf{W}+\mathbf{W}^\dagger}\, \cdot \right\rangle \tag{1.8}$$

LAB: deformed Hodge

which can be regarded as a deformation of the Hodge star

$$\star \mapsto \star \circ e^{-\mathbf{W}+\mathbf{W}^\dagger}\,. \tag{1.9}$$

This way

$$\begin{aligned} \mathbf{d}_K^{(W)} &= \mathbf{d}_K \\ \mathbf{d}^{\dagger}{}_K^{(W)} &= (\mathbf{d}_K)^{\dagger_W} = e^{\mathbf{W}+\mathbf{W}^\dagger} \circ \mathbf{d}^\dagger{}_K e^{-\mathbf{W}-\mathbf{W}^\dagger}\,. \end{aligned} \tag{1.10}$$

This suggests that to any superstring background a "relaxed" spectral triple $(\mathcal{A}, (\mathcal{H}, \langle\cdot|\cdot\rangle_W), \mathbf{d}_K \pm \mathbf{d}_K^{(W)})$ on loop space is associated, where $\mathcal{A}$ is some algebra, $(\mathcal{H}, \langle\cdot|\cdot\rangle_W)$ is the deformed Hodge inner product (1.8) and $\mathbf{d}_K \pm \mathbf{d}_K^{(W)}$ two generalized Dirac operators. This triple is "relaxed" since we won't want $\mathcal{A}$ to be C^* *and* unital, i.e. we don't want spacetime to be compact.

But "relaxed" spectral geometries following Connes can be made sense of, as is shown explicitly for the example of discrete geometry in [4]. There the above deformation approach is applied to the super*particle* limit, i.e. to differential geometry on *target* space and the deformation technique is shown to produce mimetic analogues of continuum geometry.

CIT: ForgySchreiber:2004

In the remainder of this introductory section we take a step back and indicate a broader perspective on the loop space differential geometric formulation of superstrings.

1.2 Super*gravity* as Dirac-Kähler on Wheeler-superspace

In [8] it was shown that several supersymmetric field theories admit formulations where the supersymmetry generators appear as deformed deRham operators on the system's *configuraton space*. A simple example is the free complex scalar field with its fermionic superpartner. But also for the rather more complex case of $N = 1, D =$

CIT: Schreiber:2001

$3+1$ supergravity it was shown that the supersymmetry generators Q_α (constraints, in this case) in configuration space take the form

$$Q_\alpha = e^{-W} \left\{ \begin{array}{c} \partial \\ \bar\partial \end{array} \right\} e^{W} \,, \tag{1.11}$$

LAB: sugra constraints

where W is some operator and ∂, $\bar\partial$ are holomorphic and antiholomorphic exterior derivatives on configuration space. Noting that the loop space constraints of the type II string (1.1) are really the ADM constraints of $N=1, D=1+1$ supergravity (coupled to some "matter") there seems to be a general principle at work.

In [8] the possibility of using the structure (1.11) to construct cosmological models in supergravity was analyzed, following the approach in [49]. The idea was to make a cosmological symmetry reduction in the bosonic theory thus obtaining a Wheeler-deWit constraint Laplace operator on the reduced configuration space (minisuperspace). (1.11) suggests that a corresponding cosmological model of supergravity would be obtained by promoting this Laplace operator to a Laplace-Beltrami operator (coming from a deformed exterior derivative $e^{-W} \circ \mathbf{d} \circ e^{W}$) on the exterior bundle over the original bosonic configuration space. However, it turns out nontrivial to find cosmological reductions that manifestly preserve the structure of (1.11) on minisuperspace.

CIT: Schreiber:2001
CIT: BeneGraham:1994

This problem is avoided when going to a lower dimensional form of supergravity, where the full set of contraints is tractable even without symmetry reduction. This has naturally lead to the study of superstrings discussed here.

In order to make the underlying idea transparent it is instructive to examine an even simpler case, namely supergravity in 1+0 dimensions coupled to scalar matter. This is nothing but the worldline description of what should be called the *RNS superparticle*. This shall be briefly discussed in the following subsection:

1.3 The RNS Super*particle*: Dirac-Kähler on *target space*

When taking the pointparticle limit of the type II string in the massless sector we are left with a superparticle governed by two anticommuting constraints

$$\begin{aligned} Q_1 &= \mathbf{d} + \mathbf{d}^\dagger \\ Q_2 &= i\left(\mathbf{d} - \mathbf{d}^\dagger\right) \end{aligned} \tag{1.12}$$

and their square

$$\{Q_i, Q_j\} = 2\delta_{ij}\Delta = 2\left\{\mathbf{d}, \mathbf{d}^\dagger\right\} \tag{1.13}$$

LAB: SQM algebra

on *target space*. In the RR-sector both fermionic constraints apply, so that one is dealing with RR-forms for which $\mathbf{d}\,|\psi\rangle = 0 = \mathbf{d}^\dagger\,|\psi\rangle$, while in the R-NS sector only one chiral fermionic constraint is present, $(\mathbf{d}+\mathbf{d}^\dagger)\,|\psi\rangle = 0$, which is the Dirac-Kähler equation (e.g. [50]) for fermions, i.e. the Dirac eqaution where the Dirac operator is that on the exterior bundle. Finally of course the massless bosonic vector sector is governed just by $\Delta \left|\psi\right\rangle = 0$.

CIT: BennTucker:1987

It is maybe interesting to note that the Dirac-Kähler equation was originally abandoned as a sensible model for fermions because it implies the ordinary Dirac equation only in special backgrounds, such as flat space [50]. In curved space it mixes spinor ideals on the exterior bundle and hence introduces physically inaccpetable mixing between "families" of fermions. In string theory this problem is removed by the requirement for the background fields to induce a conformal worldsheet theory, which leads to the decoupling of chiral sectors, so that this mixing does not occur (see [7] for a detailed discussion for the case of super WZW backgrounds).

CIT: BennTucker:1987

CIT: Schreiber:2003a

The above constraints of the RNS superparticle define a theory of $N = 2$ *relativistic supersymmetric quantum mechanics*, i.e. ordinary $N = 2$ SQM but with pseudo-Riemannian configuration space and with the Hamiltonian and the SUSY generators playing the roles of constraints (such relativistic SQM has been studied in some detail in [8]).

CIT: Schreiber:2001

Long ago, in [9], it was noted that such $N = 2$ SQM systems admit an interesting *deformation* which preserves the superalgebra (1.13) of the generators but introduces a *potential*.

CIT: Witten:1982

Namely one can note that in order for (1.13) to be preserved equivalently the nilpotence of $\mathbf{d} = \frac{1}{2}(Q_1 - iQ_2)$ and the star relation $\mathbf{d}^\dagger = (\mathbf{d})^\dagger$ must be preserved. This is in particular the case for conjugations of the form

$$\begin{aligned} \mathbf{d} &\mapsto e^{-W} \circ \mathbf{d} \circ e^{W} \\ \mathbf{d}^\dagger &\mapsto e^{W^\dagger} \circ \mathbf{d}^\dagger \circ e^{-W^\dagger} \,, \end{aligned} \tag{1.14}$$

for W some even graded operator.

In [9] W was chosen to be a scalar function. This adds a potential $(\nabla W)^2$ plus a fermionic term to the Hamiltonian. There it was noted that for compact Riemannian target spaces this has an intriguing relation to Morse theory. But here we shall instead be interested in simply observing that the deformation by e^W *turns on a background field* for the superparticle, which induces the potential $(\nabla W)^2$.

CIT: Witten:1982

One may immediately ask if it makes sense to study operators W other than just scalar multiplication operators. This has implicitly be done in [14, 15], where a deformation by $W = B$ a 2-form on configuration space was considered, and where it was noted that this introduces *torsion* dB. As shown in [1] and reviewed in (3.1), lifting this 2-form to loop space and deforming $\mathbf{d}_K$ there yields indeed the super-Virasoro constraints for type II strings in a Kalb-Ramon background, with the 2-form being precisely the Kalb-Ramond 2-form.

CIT: Froehlich-GrandjeanReck-nagel:1996,FroehlichGranc

CIT: Schreiber:2004

As noted in equations (38)-(42) of [**?**], it is also easy to convince oneself that there is an operator W of grade 0 which turns on a gravitational and a dilaton background.

CIT: Schreiber:Schreiber:2003a

This shows that the "Morse theory deformation" introduced in [9] is just one special case of more general deformations of the supersymmetry generators which preserve their algebra and correspond to the turning on of various background fields. CIT: Witten:1982

But the real wealth of phenomena encoded in such deformations only becomes visible when going from the superpoint particle limit back to the full superstring. This is the topic of the next subsection.

2. The RNS Super*string*: Dirac-Kähler on *loop space*

LAB: RNS on loop space

We now discuss how the superconformal algebra for the type II string can be represented and deformed in loop space.

2.1 From $N = 2, D = 1$ to $N = 1, D = 2$ along a Killing vector

The step from the superparticle to the superstring amount to enlarging the configuration space so as to incorporate the inner excitations of the string. It turns out to be convenient to construct this space in such a way that differently parametrized strings appear as different configurations. Then this space always has an isometry k whose flows are reparametrizations. We are then, following the second half of [9], looking for an exterior derivative which is nilpotent only up to reparametrizations (the following is an excerpt from [7]):

CIT: Witten:1982

CIT: Schreiber:2003a

In the presence of a Killing vector $k = k^\mu \partial_\mu$, one can consider a deformation $\mathbf{d}_k$ of the exterior derivative defined by

$$\mathbf{d}_k := \mathbf{d} + i\hat{c}_\mu k^\mu \,. \tag{2.1}$$

LAB: k-deformed exterior

The adjoint operator is then

$$\mathbf{d}^\dagger{}_k := \mathbf{d}^\dagger - i\hat{c}^\dagger{}_\mu k^\mu \,. \tag{2.2}$$

LAB: k-deformed coexteri

By the definition of the Lie-derivative one finds

$$\mathbf{d}^2 = i\mathcal{L}_k \,, \tag{2.3}$$

LAB: extdk squares to a L

and, since k is Killing, also

$$\mathbf{d}^{\dagger 2} = i\mathcal{L}_k \,. \tag{2.4}$$

LAB: coextdk squares to a

Defining

$$\begin{aligned} \mathbf{D}_{k,\pm} &= \mathbf{d}_k \pm \mathbf{d}^\dagger{}_k \\ &= \gamma^\mu_\mp \left(\hat{\nabla}_\mu \mp i k_\mu\right) \end{aligned} \tag{2.5}$$

LAB: k deformed Dirac op

one has, with $A, B \in \{+,-\}$ and $s_\pm := \pm 1$,

$$\{\mathbf{D}_{k,A}, \mathbf{D}_{k,B}\} = 2\delta_{AB}\left(s_A \mathbf{\Delta}_k + i\mathcal{L}_k\right) \,, \tag{2.6}$$

where the deformed Laplace-Beltrami operator is

$$\begin{aligned} \mathbf{\Delta}_k &:= \left\{\mathbf{d}_k, \mathbf{d}^\dagger{}_k\right\} \\ &= \mathbf{\Delta} + k^2 + i\left(\left\{\mathbf{d}^\dagger, \hat{c}_\mu k^\mu\right\} - \left\{\mathbf{d}, \hat{c}^\dagger{}_\mu k^\mu\right\}\right) \\ &= \mathbf{\Delta} + k^2 - i(\partial_{[\mu} k_{\nu]})\left(\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu} + \hat{c}^\mu \hat{c}^\nu\right) \,. \end{aligned} \tag{2.7}$$

LAB: Killing deformed La

Note that the deformed exterior differential operators still satisfy the duality relation:

$$\mathbf{d}^{\dagger}{}_{k} = -\bar{\star}\,\mathbf{d}_{k}\,\bar{\star}\,. \tag{2.8}$$

This gives us the algeba of *global* $D = 2$, $N = 1$ supersymmetry. The local algebra is obtained by considering appropriate modes on the string: (The following is an excerpt from [1])

CIT:
Schreiber:2004

2.2 Local K-deformed deRham operators on parametrized loop space

LAB: Local
K-deformed
dRham operators
on parametrized
loop space

As was discussed above, one may obtain from the exterior derivative and its adjoint on a manifold the generators of a global $D = 2$, $N = 1$ superalgebra by deforming with a Killing vector. The generic Killing vector field on loop space is the reparametrization generator $K^{(\mu,\sigma)} = X'^{\mu}(\sigma)$. Using this to deform the exterior derivative and its adjoint as in equation (19) of [7] yields the operators

CIT:
Schreiber:2003a

$$\begin{aligned}
\mathbf{d}_K &:= \mathbf{d} + i\mathcal{E}_{(\mu,\sigma)}X'^{(\mu,\sigma)} \\
\mathbf{d}^{\dagger}{}_K &:= \mathbf{d}^{\dagger} - i\mathcal{E}^{\dagger}_{(\mu,\sigma)}X'^{(\mu,\sigma)}\,,
\end{aligned} \tag{2.9}$$

LAB: k-defomred extds o

(where for convenience we set $T = 1$ for the moment) which generate a *global* superalgebra. Before having a closer look at this algebra let us first enlarge it to a local superalgebra by considering the *modes* defined by

$$\begin{aligned}
\mathbf{d}_{K,\xi} &:= \left[\mathcal{N}_\xi, \mathbf{d}^{*}_{K}\right] \\
\mathbf{d}^{\dagger}{}_{K,\xi^*} &:= -\left[\mathcal{N}_\xi, \mathbf{d}^{\dagger}{}^{*}_{K}\right]\,,
\end{aligned} \tag{2.10}$$

LAB: super-virasoro gener

where $\cdot^*$ is the complex adjoint and $\mathcal{N}_\xi$ is the ξ-mode of the form number operator. They explicitly read

$$\begin{aligned}
\mathbf{d}_{K,\xi} &= \int d\sigma\; \xi(\sigma)\left(\mathcal{E}^{\dagger\mu}(\sigma)\,\partial^{c}_{\mu}(\sigma) + i\mathcal{E}_{\mu}(\sigma)\,X'^{\mu}(\sigma)\right) \\
\mathbf{d}^{\dagger}{}_{K,\xi} &= -\int d\sigma\; \xi(\sigma)\left(\mathcal{E}^{\mu}(\sigma)\,\nabla_{\mu}(\sigma) + i\mathcal{E}^{\dagger}_{\mu}(\sigma)\,X'^{\mu}(\sigma)\right)\,.
\end{aligned} \tag{2.11}$$

LAB: modes of deformed

Making use of the fact that $\mathbf{d}_{K,\xi}$ is actually independent of the background metric, it is easy to establish the algebra of these operators. We do this for the "classical" fields, ignoring normal ordering effects and the anomaly:

The anticommutator of the operators (2.10) with themselves defines the ξ-mode $\mathcal{L}_{K,\xi}$ of the Lie-derivative $\mathcal{L}_K$ along K:

$$\left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}_{K,\xi_2}\right\} = 2i\mathcal{L}_{K,\xi_1\xi_2}\,, \tag{2.12}$$

LAB: anticom of extds

where

$$\mathcal{L}_\xi = \int d\sigma\; \left(\xi(\sigma)\,X'^{\mu}(\sigma)\,\partial^{c}_{\mu}(\sigma) + \sqrt{\xi}\left(\sqrt{\xi}\mathcal{E}^{\dagger\mu}\right)'(\sigma)\,\mathcal{E}_{\mu}(\sigma)\right)\,. \tag{2.13}$$

We say that a field $A(\sigma)$ has *reparametrization weight* w if

$$\begin{aligned} \left[\mathcal{L}_\xi, A(\sigma)\right]_\iota &= (\xi A' + w\xi' A)(\sigma) \\ \left[\mathcal{L}_{\xi_1}, A_{\xi_2}\right]_\iota &= A_{(w-1)\xi_1'\xi_2 - \xi_1\xi_2'}\,, \end{aligned} \tag{2.14}$$

LAB: transformation unde

where $A_\xi := \int d\sigma\, \xi A$. For the basic fields we find

$$\begin{aligned} w(X^\mu) &= 0 \\ w(X'^\mu) &= 1 \\ w\big(\partial^c_\mu\big) &= 1 \\ w(\Gamma^\mu_\pm) &= 1/2\,. \end{aligned} \tag{2.15}$$

LAB: reparametrization w

Because of $w(AB) = w(A)+w(B)$ it follows that $\mathbf{d}_{K,\xi}$ and $\mathbf{d}^\dagger{}_{K,\xi}$ are modes of integrals over densities of reparametrization weight $w = 3/2$. This implies in particular that

$$\left[\mathcal{L}_{\xi_1}, \mathbf{d}_{K,\xi_2}\right] = \mathbf{d}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')} \tag{2.16}$$

LAB: comm of L extd

$$\left[\mathcal{L}_{K,\xi_1}, \mathcal{L}_{K,\xi_2}\right] = \mathcal{L}_{K,(\xi_1'\xi_2 - \xi_1\xi_2')}\,. \tag{2.17}$$

By taking the adjoint of (2.12) and (2.16) (or by doing the calculation explicitly), analogous relations are found for $\mathbf{d}^\dagger{}_{K,\xi}$:

$$\begin{aligned} \left\{\mathbf{d}^\dagger{}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} &= 2i\mathcal{L}_{K,\xi_1\xi_2} \\ \left[\mathcal{L}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right] &= \mathbf{d}^\dagger{}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')}\,. \end{aligned} \tag{2.18}$$

LAB: comm L coextd

Equations (2.12), (2.16), and (2.18) give us part of the sought-after algebra. A very simple and apparently unproblematic but rather crucial step for finding the rest is to now define the *modes of the deformed Laplace-Beltrami operator* as the right hand side of

$$\left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = \mathbf{\Delta}_{K,\xi_1\xi_2}\,. \tag{2.19}$$

LAB: extdxi coextdxi anti

For this definition to make sense one needs to check that

$$\left\{\mathbf{d}_{K,\xi_1\xi_3}, \mathbf{d}^\dagger{}_{K,\xi_2}\right\} = \left\{\mathbf{d}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2\xi_3}\right\}\,. \tag{2.20}$$

LAB: crucial condition

It is easy to verify that this is indeed true for the operators as given in (2.11). However, in §**??** (p.**??**) it is found that this condition is a rather strong constraint on the admissible perturbations of these operators, and the innocent looking equation (2.20) plays a pivotal role in the algebraic construction of superconformal field theories in the present context.

With $\mathbf{\Delta}_{K,\xi}$ consistently defined as in (2.19) all remaining brackets follow by using the Jacobi-identity:

$$\left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \mathbf{d}_{K,\xi_2}\right] = i\mathbf{d}^\dagger{}_{K,(\frac{1}{2}\xi_1'\xi_2 - \xi_1\xi_2')}$$

$$\begin{aligned}
\left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \mathbf{d}^\dagger{}_{K,\xi_2}\right] &= i\mathbf{d}_{K,(\frac{1}{2}\xi_1'\xi_2-\xi_1\xi_2')} \\
\left[\frac{1}{2}\mathbf{\Delta}_{K,\xi_1}, \frac{1}{2}\mathbf{\Delta}_{K,\xi_2}\right] &= -\mathcal{L}_{K,(\xi_1'\xi_2-\xi_1\xi_2')}\,. \qquad (2.21)
\end{aligned}$$

LAB: remaining brackets

In order to make the equivalence to the super-Virasoro algebra of the algebra thus obtained more manifest consider the modes of the K-deformed Dirac-Kähler operators on loop space:

$$\begin{aligned}
\mathbf{D}_{K,\pm} &:= \mathbf{d}_K \pm \mathbf{d}^\dagger{}_K \\
&= \Gamma_{\mp}^{(\mu,\sigma)}\left(\hat{\nabla}_{(\mu,\sigma)} \mp iX'_{(\mu,\sigma)}\right) \\
\mathbf{D}_{K,\pm,\xi} &:= \mathbf{d}_{K,\xi} \pm \mathbf{d}^\dagger{}_{K,\xi}\,. \qquad (2.22)
\end{aligned}$$

LAB: K-deformed loop sp

They are the supercharges which generate the super-Virasoro algebra in the usual chiral form

$$\begin{aligned}
\{\mathbf{D}_{K,\pm,\xi_1}, \mathbf{D}_{K,\pm,\xi_2}\} &= 4\left(\pm\frac{1}{2}\mathbf{\Delta}_{\xi_1\xi_2} + i\mathcal{L}_{\xi_1\xi_2}\right) \\
\left[\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1} + i\mathcal{L}_{\xi_1}, \mathbf{D}_{K,\pm,\xi_2}\right] &= 2\mathbf{D}_{K,\pm,\frac{1}{2}\xi_1'\xi_2-\xi_1\xi_2'} \\
\left[\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1} + i\mathcal{L}_{\xi_1}, \pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_2} + i\mathcal{L}_{\xi_2}\right] &= 2i\left(\pm\frac{1}{2}\mathbf{\Delta}_{K,\xi_1'\xi_2-\xi_1\xi_2'} + i\mathcal{L}_{\xi_1'\xi_2-\xi_1\xi_2'}\right) \qquad (2.23)
\end{aligned}$$

LAB: suVir algebra in alm

It is easily seen that this acquires the standard form when we set $\xi(\sigma) = e^{in\sigma}$ for $n \in \mathbb{N}$. In order to make the connection with the usual formulation more transparent consider a flat target space. If we define the operators

$$\mathcal{P}_{\pm,(\mu,\sigma)} := \frac{1}{\sqrt{2T}}\left(-i\partial_{(\mu,\sigma)} \pm TX'_{(\mu,\sigma)}\right) \qquad (2.24)$$

LAB: functional definition

with commutator

$$\left[\mathcal{P}_{A,(\mu,\sigma)}, \mathcal{P}_{B,(\nu,\sigma')}\right] = is_A\delta_{AB}\eta_{\mu\nu}\delta'_{\sigma,\sigma'}\,, \qquad \text{for } g_{\mu\nu} = \eta_{\mu\nu} \qquad (2.25)$$

LAB: the functional comn

we get, up to a constant factor, the usual modes

$$\begin{aligned}
\mathbf{D}_{K,\pm,\xi} &= \sqrt{2}i\int d\sigma\xi(\sigma)\,\Gamma^\mu_\mp(\sigma)\,\mathcal{P}_{\mu,\mp}(\sigma) \\
\mathbf{D}^2_{K,\pm,\xi^2} &= \pm 2\int d\sigma\ \left(2\xi^2(\sigma)\,\mathcal{P}_\mp(\sigma)\cdot\mathcal{P}_\mp(\sigma) + \xi(\sigma)\,(\xi\Gamma_\mp)'(\sigma)\cdot\Gamma_\mp(\sigma)\right)\,. \qquad (2.26)
\end{aligned}$$

2.3 Supervirasoro and its deformations in polar form

Consider some realization of the superconformal generators $G_r, \bar{G}_r, T_n, \bar{T}_n$ of the type II superstring. We are looking for consistent deformations of these operators to operators $G_r^\Phi, \bar{G}_r^\Phi, T_n^\Phi, \bar{T}_n^\Phi$ which still satisfy the superconformal algebra and so that the generator of spatial worldsheet reparametrizations remains invariant:

$$L_n^\Phi - \bar{L}_{-n}^\Phi \overset{!}{=} L_n - \bar{L}_{-n} \,. \tag{2.27}$$

LAB: invariance of spatial

This condition follows from a canonical analysis of the worldsheet action, which is nothing but 1+1 dimensional supergravity coupled to various matter fields. As for all gravitational theories, their ADM constraints break up into spatial diffeomorphism constraints as well as the Hamiltonian constraint, which alone encodes the dynamics.

The condition (2.27) can also be understood in terms of boundary state formalism. As discussed below, the operator $\mathcal{B}$ related to a nontrivial bounday state $|\mathcal{B}\rangle$ can be interpreted as inducing a deformation $G_r^\Phi := \mathcal{B}^{-1} G_r \mathcal{B}$, etc.

In any case, we are looking for isomorphisms of the superconformal algebra which satisfy (2.27):

To that end, let d_r and $d_r^\dagger$ be the modes of the polar combinations of the left- and right-moving suppercurrents

$$\begin{aligned} d_r &:= G_r + i\bar{G}_{-r} \\ d_r^\dagger := (d_r)^\dagger &= G_r - i\bar{G}_{-r} \,. \end{aligned} \tag{2.28}$$

LAB: polar supercurrent r

These are the 'square roots' of the reparameterization generator

$$\mathcal{L}_n := -i \left(L_n - \bar{L}_{-n} \right) \,, \tag{2.29}$$

LAB: rep gen

i.e.

$$\{d_r, d_s\} = \{d_r^\dagger, d_s^\dagger\} = 2i\mathcal{L}_{r+s} \,. \tag{2.30}$$

Under a deformation the right hand side of this equation must stay invariant (2.27) so that

$$\begin{aligned} d_r^\Phi &:= d_r + \Delta_\Phi d_r \\ d_r^{\dagger\Phi} &:= d_r^\dagger + (\Delta_\Phi d_r)^\dagger \end{aligned} \tag{2.31}$$

implies that the shift $\Delta_\Phi d_r$ of d_r has to satisfy

$$\{d_r, \Delta_\Phi d_s\} + \{d_s, \Delta_\Phi d_r\} + \{\Delta_\Phi d_r, \Delta_\Phi d_s\} = 0 \,. \tag{2.32}$$

LAB: shift in loop space e

One large class of solutions of this equation is

$$\Delta_\Phi d_r = A^{-1} \left[d_r, A \right] \,, \qquad \text{for } [\mathcal{L}_n, A] = 0 \;\forall\, n \,, \tag{2.33}$$

LAB: similarity transform

where A is any even graded operator that is spatially reparameterization invariant, i.e. which commutes with (2.29).

When this is rewritten as

$$\begin{aligned} d_r^{\Phi} &= A^{-1} \circ d_r \circ A \\ d_r^{\dagger\Phi} &= A^{\dagger} \circ d_r^{\dagger} \circ A^{\dagger -1} \end{aligned} \tag{2.34}$$

LAB: simtrafo on susy cu

one sees explicitly that the formal structure involved here is a direct generalization of that used in [9] in the study of the relation of deformed generators in supersymmetric quantum *mechanics* to Morse theory. Here we are are concerned with the direct generalization of this mechanism from $1+0$ to $1+1$ dimensional supersymmetric field theory.

CIT: Witten:1982

In $1+0$ dimensional SQFT (i.e. supersymmetric quantum mechanics) relation (2.34) is sufficient for the deformation to be truly an isomorphism of the algebra of generators. In 1+1 dimensions, on the superstring's worldsheet, there is however one further necessary condition for this to be the case. Namely the (modes of the) new worldsheet Hamiltonian constraint $H_n = L_n + \bar{L}_-n$ must clearly be defined as

$$H_n^{\Phi} := \frac{1}{2}\left\{d_r^{\Phi}, d_{n-r}^{\dagger\Phi}\right\} - \delta_{n,0}\frac{c}{12}\left(4r^2-1\right) \tag{2.35}$$

LAB: def deformed worlds

and (2.32) alone does not guarantee that this is *unique* for all $r \neq n/2$. *If* it is, however, then the Jacobi identity already implies that

$$\begin{aligned} G_r^{\Phi} &:= \frac{1}{2}\left(d_r^{\Phi} + d_r^{\dagger\Phi}\right) \\ L_n^{\Phi} &:= \frac{1}{4}\left(\left\{d_r^{\Phi}, d_{n-r}^{\dagger\Phi}\right\} + \left\{d_r^{\Phi}, d_{n-r}^{\Phi}\right\}\right) - \delta_{r,n/2}\frac{c}{24}\left(4r^2-1\right) \\ \bar{G}_r^{\Phi} &:= -\frac{i}{2}\left(d_{-r}^{\Phi} - d_{-r}^{\dagger\Phi}\right) \\ L_n^{\Phi} &:= \frac{1}{2}\left(\left\{d_{-r}^{\Phi}, d_{r-n}^{\dagger\Phi}\right\} - \left\{d_{-r}^{\Phi}, d_{r-n}^{\Phi}\right\}\right)\,, \qquad \forall\, r \neq n/2 \end{aligned} \tag{2.36}$$

generate two mutually commuting copies of the super Virasoro algebra.

In order to see this first note that the two copies of the unperturbed Virasoro algebra in terms of the 'polar' generators $d_r, d_r^{\dagger}, i\mathcal{L}_m, H_m$ read

$$\begin{aligned} \{d_r, d_s\} &= 2\, i\mathcal{L}_{r+s} = \left\{d_r^{\dagger}, d_s^{\dagger}\right\} \\ [i\mathcal{L}_m, d_r] &= \frac{m-2r}{2} d_{m+r} \\ \left[i\mathcal{L}_m, d_r^{\dagger}\right] &= \frac{m-2r}{2} d_{m+r}^{\dagger} \\ [i\mathcal{L}_m, i\mathcal{L}_n] &= (m-n) i\mathcal{L}_{m+n} \\ [i\mathcal{L}_m, H_n] &= (m-n) i\mathcal{H}_{m+n} + \frac{c}{6}(m^3-m)\delta_{m,-n} \\ [H_m, d_r] &= \frac{m-2r}{2} d_{m+r}^{\dagger} \\ \left[H_m, d_r^{\dagger}\right] &= \frac{m-2r}{2} d_{m+r} \\ [H_m, H_n] &= (m-n)\, i\mathcal{L}_{m+n}\,. \end{aligned} \tag{2.37}$$

Now check that these relations are obeyed also by the deformed generators d_r^Φ, $d_r^{\dagger\Phi}$, $i\mathcal{L}_m$, H_m^Φ using the two conditions (2.34) and (2.35):

First of all the relations

$$\begin{aligned}\left[i\mathcal{L}_m, d_r^\Phi\right] &= \frac{m-2r}{2} d_{m+r}^\Phi \\ \left[i\mathcal{L}_m, d_r^{\dagger\Phi}\right] &= \frac{m-2r}{2} d_{m+r}^{\dagger\Phi}\end{aligned} \tag{2.38}$$

LAB: bracket Lm dr

follow simply from (2.34) and the original bracket $[L_m, G_r] = \frac{m-2r}{2} G_{m+r}$ and immediately imply

$$[i\mathcal{L}_m, i\mathcal{L}_n] = (m-n) i\mathcal{L}_{m+n} \tag{2.39}$$

(note that here the anomaly of the left-moving sector cancels that of the right-moving one).

Furthermore

$$\begin{aligned}\left[i\mathcal{L}_m, H_n^\Phi\right] &= \left[i\mathcal{L}_m, \frac{1}{2}\left\{d_r^\Phi, d_{n-r}^{\dagger\Phi}\right\}\right] \\ &\overset{(2.38)}{=} \frac{m-2r}{4}\left\{d_{m+r}^\Phi, d_{n-r}^{\dagger\Phi}\right\} + \frac{m-2(n-r)}{4}\left\{d_r^\Phi, d_{m+n-r}^{\dagger\Phi}\right\} \\ &\overset{(2.35)}{=} (m-n)H_{m+n}^\Phi + \delta_{m,-n}\frac{c}{6}\left(\frac{m-2r}{4}(4(m+r)^2-1) + \frac{m-2(n-r)}{4}(4r^2-1)\right) \\ &= (m-n)H_{m+n}^\Phi + \delta_{m,-n}\frac{c}{6}\left(m^3-m\right) \,.\end{aligned} \tag{2.40}$$

(Here the anomalies from both sectors add.)

The commutator of the Hamiltonian with the supercurrents is obtained for instance by first writing:

$$\begin{aligned}\left[H_m^\Phi, d_r^\Phi\right] &= \frac{1}{2}\left[\left\{d_r^\Phi, d_{m-r}^{\dagger\Phi}\right\}, d_r^\Phi\right] \\ &= -\frac{1}{2}\left[\left\{d_r^\Phi, d_r^\Phi\right\}, d_{m-r}^{\dagger\Phi}\right] - \frac{1}{2}\left[\left\{d_r^\Phi, d_{m-r}^{\dagger\Phi}\right\}, d_r^\Phi\right] \\ &= -\left[i\mathcal{L}_{2r}, d_{m-r}^{\dagger\Phi}\right] - \left[H_m^\Phi, d_r^\Phi\right] \\ &= (m-2r)d_{m+r}^{\dagger\Phi} - \left[H_m^\Phi, d_r^\Phi\right] \,,\end{aligned} \tag{2.41}$$

from which it follows that

$$\left[H_m^\Phi, d_r^\Phi\right] = \frac{(m-2r)}{2} d_{m+r}^{\dagger\Phi} \tag{2.42}$$

and similarly

$$\left[H_m^\Phi, d_r^{\dagger\Phi}\right] = \frac{(m-2r)}{2} d_{m+r}^\Phi \,. \tag{2.43}$$

This can finally be used to obtain

$$\left[H_m^\Phi, H_n^\Phi\right] = (m-n) i\mathcal{L}_{m+n}\,. \tag{2.44}$$

In summary this shows that every operator A which

1. commutes with $i\mathcal{L}_m$
2. is such that $\left\{A^{-1} d_r A, A^\dagger d^\dagger_{n-r} A^{\dagger -1}\right\} - \delta_{n,0}\frac{c}{12}(4r^2-1)$ is *independent* of r

defines a consistent deformation of the super Virasoro generators and hence a string background which satisfies the classical equations of motion of string field theory.

In [1] it was shown how at least all massless NS and NS-NS backgrounds of the closed string can be obtained by deformations A of the form $A = e^{\mathbf{W}}$, where $\mathbf{W}$ is related to the vertex operator of the respective background field. Furthermore, it was demonstrated in [7] that the structure (2.34) of the SCFT deformations allows to handle superstring evolution in nontrivial backgrounds as generalized Dirac-Kähler evolution in loop space.

CIT: Schreiber:2004

CIT: Schreiber:2003a

In the special case where A is *unitary* the similarity transformations (2.34) of d and $d^\dagger$ and hence of all other elements of the super-Virasoro algebra are identical and the deformation is nothing but a unitary transformation. It was discussed in [1] that gauge transformations of the background fields, such as reparameterizations or gauge shifts of the Kalb-Ramond field are described by such unitary transformation.

CIT: Schreiber:2004

In particular, an *abelian* gauge field background was shown to be induced by the Wilson line

$$\mathbf{W}^{(A)} = i \oint d\sigma\; A_\mu(X(\sigma))\, X'^\mu(\sigma) \tag{2.45}$$

LAB: abelian gauge field

of the gauge field along the closed string.

By using boundary state formalism these facts nicely generalize to the *open* string. When the closed string is 'cut open' we can apply the respective unitary transformation to the boundary state. From a boundary state $|\alpha\rangle$ describing a bare brane we should hence get the boundary state of that brane with a gauge field turned on by writing $U\,|\alpha\rangle$, where U is the Wilson line of that gauge field along the closed string at the boundary. Indeed, this was shown to be the correct boundary state describing open strings in gauge field backgrounds in [26, 24, 25].

CIT: MaedaNakatsuOonishi:2004,Hashimoto:2000

2.4 Calculus on loop space

The loop space calculus used above can be given a clean formulation by applying it to loop space functions that are single path-ordered integrals over loops and then formally extending this action to formal series of such path ordered integrals. Aspects of these constructions can be found discussed for instance in [51, 31, 52].

CIT: Rajeev:2004,Hofman:2002,G

3. Applications

LAB: applications

3.1 Deformations and NCG

LAB: Deformations and NCG

3.2 Hamiltonian evolution in loop space

3.3 Boundary states

3.3.1 Wilson lines along the string

3.3.2 Pohlmeyer invariants / boundary state generators

3.4 Loop space and 1-gerbes: Nonabelian 2-form connections

4. Conclusion and Outlook

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