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Notes on OSFT

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Abstract: Private notes on open string field theory concerned with the question how classical solutions of OSFT correspond to deformations of worldsheet CFTs.

Contents

LAB: -TOC-

1. Introduction

This are private notes on some aspects of string field theory with an eye on the question how deformations of worldsheet CFTs can be understood from the point of view of classical solutions of string field theory, and vice versa. For the most part well known material is reviewed. but also some first steps in the direction of associating classical string field backgrounds with deformations of the (super) Virasoro generators are indicated (§2.4.2 (p.10), §3.4 (p.21), and §4.2.2 (p.29)).

An introduction to basic concepts needed in string field theory is given in the appendices §A (p.37) (worldsheet BRST methods), §B (p.44) (string fields and the star product).

2. Bosonic String field theory

2.1 Definitions and notation

Consider cubic open bosonic string field theory (as for instance reviewed in [1]) with Q the BRST operator corresponding to the trivial Minkowski background. CIT: Ohmori:2001

The crucial algebraic relations are

- nilpotence: $Q^2 = 0$
- graded Leibnitz rule:

$$Q(A * B) = (QA) * B + (-1)^{|A|} * (QB) \tag{2.1}$$

LAB: graded Leibnitz

- Stokes' law:

$$\int QA = 0 \tag{2.2}$$

- graded trace property of the integral:

$$\int A * B = (-1)^{|A||B|} \int B * A \tag{2.3}$$

The action of a string field Φ of unit ghost number is

$$S[\Phi] = \int \Phi * Q\Phi + \frac{2}{3} \int \Phi * \Phi * \Phi \tag{2.4}$$

LAB: bosonic cubic OSFT

(where we have set the string coupling to unity for notational convenience) which is invariant under the gauge tranformation

$$\delta\Phi = Q\Lambda + \Phi * \Lambda - \Lambda * \Phi \tag{2.5}$$

LAB: gauge trafo

for arbitrary Λ of 0 ghost number.

The classical equations of motion are hence

$$Q\Phi + \Phi * \Phi = 0\,. \tag{2.6}$$

LAB: string field equation

2.2 Background deformations

LAB: shifting backgrounds

The following reviews some general concepts which for the most part have been mentioned in [2].

CIT: Horowitz-LykkenRohmStrominger:1986

Consider a string field relative to a solution Φ_0 of the classical equations of motion (2.6):

$$\Phi := \Phi_0 + \psi \,. \tag{2.7}$$

LAB: shifted field

The action can now be decomposed as

$$S[\Phi] = S[\Phi_0] + S_{\Phi_0}[\psi] \tag{2.8}$$

with

$$S_{\Phi_0}[\psi] = \int \psi * (Q + 2\Phi_0 *)\, \psi + \frac{2}{3} \int \psi * \psi * \psi \,, \tag{2.9}$$

LAB: relative string field a

where the terms linear in ψ vanish due to Φ_0 being a classical solution.

Keeping Φ_0 fixed and varying only ψ (which is just a reparameterization of the space of string fields and hence completely general) one obtains the equation of motion in terms of ψ:

$$(Q + [\Phi_0 *, \cdot])\, \psi + \psi * \psi = 0 \,. \tag{2.10}$$

LAB: shifted EOM

(Here the brackets denote the *graded* commutator.) This of course also follows from inserting (2.7) into (2.6).

This has essentially the same form as the equivalent equation of motion (2.6) for the full field Φ except for a modification of the BRST operator. In order to make this more transparent introduce the notion of *string field multiplication operators* $\hat{A}$ defined by

$$\hat{A}\, |B\rangle := |A * B\rangle \,. \tag{2.11}$$

For convenience define the integral over such a multiplication operator to be equal to the integral over the corresponding field:

$$\int \hat{\Phi} := \int \Phi \,. \tag{2.12}$$

Note that, due to (2.1), we have

$$\left[Q, \hat{A}\right] = \widehat{QA} \,, \tag{2.13}$$

where the products on the left are understood to be operator products.

Using this notation (2.9) is rewritten as

$$S_{\Phi_0}[\psi] = \int \hat{\psi} \left[\left(Q + \hat{\Phi}_0\right), \hat{\psi}\right] + \frac{2}{3} \int \hat{\psi}\hat{\psi}\hat{\psi} \,, \tag{2.14}$$

which suggests to identify graded commutation with $Q + \Phi_0$ as a new BRST operator. Indeed, it has all the necessary formal properties:

- nilpotence

$$\begin{aligned}\left[\left(Q+\hat{\Phi}_0\right),\left[\left(Q+\hat{\Phi}_0\right),\hat{A}\right]\right] &= \left[\left(Q+\hat{\Phi}_0\right)^2,\hat{A}\right]\\ &= \left[\widehat{Q\Phi_0}+\hat{\Phi}_0\hat{\Phi}_0,\hat{A}\right]\\ &= 0\end{aligned} \tag{2.15}$$

- graded Leibnitz property

$$\left[Q+\hat{\Phi}_0,\hat{A}\hat{B}\right] = \left[Q+\hat{\Phi}_0,\hat{A}\right]\hat{B}+(-1)^{|A|}\hat{A}\left[Q+\hat{\Phi}_0,\hat{B}\right] \tag{2.16}$$ LAB: new graded Leibnitz

- Stokes' law

$$\int\left[\left(Q+\hat{\Phi}_0\right),\hat{A}\right] = 0\,, \tag{2.17}$$ LAB: new Stokes

which implies in particular that graded partial integration works as usual, too:

$$\int A\left[\left(Q+\hat{\Phi}\right),B\right] = -(-1)^{|A|}\int\left[\left(Q+\hat{\Phi}\right),A\right]B\,. \tag{2.18}$$

All this suggests that

$$Q^{\Phi_0}:|\psi\rangle\mapsto Q\,|\psi\rangle+|\Phi_0*\psi\rangle-(-1)^{|\psi|}\,|\psi*\Phi_0\rangle \tag{2.19}$$ LAB: new BRST operator

is the worldsheet BRST operator for open bosonic strings in the background described by the background field Φ.

Literature:

The above mechanism of backgound shifts and BRST operator deformations plays a role in searches for *vacuum string field theory* (VSFT), e.g. section 5 of [3]. CIT: KishimotoOhmori:2002

We may formally restrict the above considerations to the case where $Q=0$ identically vanishes. In this case we are left with just the interaction vertex $S\to\frac{2}{3}\int\Phi*\Phi*\Phi$. All the above goes through as before and we find that relative to a classical solution of the equations of motion, which now read $\Phi_0*\Phi_0=0$, the relative field ψ sees a nontrivial BRST operator $Q=[\Phi_0*,\cdot]$.

This is precisely the old insight of [4]. There it was furthermore partially shown that there is a Φ_0 which reproduces the BRST operator for Minkowski background this way.[1] CIT: Hata:1986

The modern form of this purely cubic action is described in [2] using the concepts described in §2.4 (p.8) CIT: Horowitz-LykkenRohmStrominger:1986

2.3 Gauge transformations

LAB: gauge transformations

Now consider what happens to the BRST operator as gauge transformations on the background string field Φ_0 are performed:

[1]For comparison note that the graded bracket may often be omitted in the action due to $\int\hat{\psi}[\hat{\Phi},\hat{\psi}]=2\int\hat{\psi}\hat{\Phi}\hat{\psi}$.

Start with the trivial solution $\psi = 0$ to (2.10) and perform a gauge transformation parameterized by a string field Λ of 0 ghost number: $\psi \to \psi + \epsilon\delta\psi = \epsilon\left[\tilde{Q}, \hat{\Lambda}\right]$.

Due to the gauge invariance of the action this is still a solution to first order in ϵ.

We may hence apply the reasoning of §(2.2) again by parameterizing the string field now as $\Phi = \Phi_0 + \psi + \psi^{(1)}$. Relative to (2.9) one now obtains the action:

$$S_\psi[\psi^{(1)}] = \int \hat{\psi}^{(1)} \left(\tilde{Q} + \epsilon\left[\tilde{Q}, \hat{\Lambda}\right]\right) \hat{\psi}^{(1)} + \frac{2}{3}\int \hat{\psi}^{(1)}\hat{\psi}^{(1)}\hat{\psi}^{(1)} . \tag{2.20}$$

(Here $\tilde{Q} = Q + \Phi_0 *$ is the BRST operator corresponding to the original shift to the classical solution Φ_0.)

In accordance with the general result of §2.2 the gauge transformation on the background Φ_0 yields a new BRST operator

$$\tilde{Q}^{(1)} := \tilde{Q} + \epsilon\left[\tilde{Q}, \hat{\Lambda}\right] . \tag{2.21}$$

LAB: gauge trafo on BRS

Note that

$$\left[\tilde{Q}, \hat{\Lambda}\right] = (Q\Lambda + \Phi_0 * \Lambda - \Lambda * \Phi_0)\hat{} \tag{2.22}$$

as it should be according to (2.5).

But the point here is that by re-iterating the above process (2.21) one obtains a sequence $\left\{\tilde{Q}^{(n)}\right\}_n$ of BRST operators:

$$\tilde{Q}^{(n+1)} := \tilde{Q}^{(n)} + \epsilon\left[\tilde{Q}^{(n)}, \hat{\Lambda}\right] . \tag{2.23}$$

In the limit of finite gauge transformations this yields

$$\lim_{\epsilon\to 0} \tilde{Q}^{([1/\epsilon])} = \exp\left(-\hat{\Lambda}\right)\tilde{Q}\exp\left(\hat{\Lambda}\right) . \tag{2.24}$$

Together with §2.2 this should imply that the gauge transformations of the background fields of string theory manifest themselves as conjugations of the Virasoro generators on the worldsheet by the exponentiated (with respect to the star product) generators of the gauge transformation.

This should make good sense due to the fact that the string field action is formally that of Chern-Simons theory, Q is formally the ordinary exterior derivative $\mathbf{d}$, Φ the gauge connection $\mathbf{A}$ and $[Q + \Phi *, \cdot]$ the gauge covariant derivative $\mathbf{d}_A$ which under gauge transformations does transform as $\mathbf{d}_A \to g^{-1} \circ \mathbf{d}_A \circ g$.

As an operator on the string's Hilbert space with the scalar product

$$\langle\Psi|\Phi\rangle = \int \Psi * \Phi \tag{2.25}$$

(where the equality holds on the subspace that satisfies the *reality condition* (2.8) [1], *cf.* (168) [5]) the operator $\left[\hat{\Lambda}, \cdot\right]$ is anti-Hermitean

CIT: Ohmori:2001

CIT: TaylorZwiebach:2004

$$\langle\Psi|\left[\hat{\Lambda}, \cdot\right]\Phi\rangle = -\langle\left[\hat{\Lambda}, \cdot\right]\Psi|\Phi\rangle \tag{2.26}$$

due to the fact that Λ is of ghost number 0. It hence follows that $\exp\left[\hat{\Lambda},\cdot\right]$ is in fact unitary.

In other words, a gauge transformation of the background fields manifests itself as a unitary transformation

$$\begin{aligned} |\psi\rangle &\mapsto \exp\Big(\left[\hat{\Lambda},\cdot\right]\Big)\,|\psi\rangle \\ Q &\mapsto \exp\Big(\left[\hat{\Lambda},\cdot\right]\Big)\circ Q\circ \exp\Big(-\left[\hat{\Lambda},\cdot\right]\Big) \end{aligned} \tag{2.27}$$

on the string's Hilbert space.[2]

This fact has been used in particular in [6] for finding new worldsheet theories from deformed BRST operators. CIT: Giannakis:2002

[2] A simple but potentially confusing point is that when used inside the graded commutator we have $[Q,\cdot]\mapsto\left[e^{\hat{\Lambda}}\circ Q\circ e^{-\hat{\Lambda}},\cdot\right]$ while as a standalone operator indeed $Q\mapsto e^{[\hat{\Lambda},\cdot]}\circ Q\circ e^{-[\hat{\Lambda},\cdot]}$. Indeed, both yield the same result, namely

$$\left[e^{\hat{\Lambda}}\circ Q\circ e^{-\hat{\Lambda}},\hat{\psi}\right] = \widehat{Q\psi}+\left[e^{\hat{\Lambda}}\circ\left[Q,e^{-\hat{\Lambda}}\right],\hat{\psi}\right] \tag{2.28}$$

and

$$\begin{aligned} e^{[\hat{\Lambda},\cdot]}Qe^{-[\hat{\Lambda},\cdot]}\,|\psi\rangle &= e^{[\hat{\Lambda},\cdot]}Q\;\; e^{-\hat{\Lambda}}\psi e^{\hat{\Lambda}}\Big\rangle \\ &= Q\,|\psi\rangle + \;\left[e^{\hat{\Lambda}}\circ\left[Q,e^{-\hat{\Lambda}}\right],\hat{\psi}\right]\Big\rangle\,. \end{aligned} \tag{2.29}$$

LAB: finite gauge trafo

2.4 CFTs from string field backgrounds

LAB: CFTs from string field backgrounds

It is generally assumed that every classical solution Φ_0 to the string field equations of motion (2.6) corresponds to a new conformal field theory on the worldsheet which is associated with the deformed BRST operator $\tilde{Q} = Q + [\Phi *, \cdot]$.

This was mentioned parenthetically in [7] (p. 256) and worked out in detail for infinitesimal shifts in [8, 9], which is briefly reviewed in section 2.4 of [10]. The extension to finite deformations of a certain type has recently been considered in [11] for the open bosonic string, using techniques developed in [12] in the context of superstring field theory.

CIT: Witten:1986
CIT: Sen:1990,Sen:1990b
CIT: SenZwiebach:1993
CIT: Kluson:2003
CIT: Kluson:2002a

A crucial insight in the latter papers is how the graded star commutator with some string field of unit weight translates to an operator in the string's Hilbert space:

2.4.1 The graded star commutator

So let, for the open string, $W(z)$ be a chiral field of unit weight on the upper half plane and let

$$\mathrm{W}(z) := \begin{cases} W(z) & \text{for } \mathrm{Im}(z) \geq 0 \\ \Omega \bar{W}(z) & \text{for } \mathrm{Im}(z) < 0 \end{cases} \tag{2.30}$$

be its continuation to the entire complex plane, where Ω is the gluing map (e.g. section 2.2. of [13]).

CIT: RecknagelSchomerus:1999

The contour integral

$$\mathbf{W} := \oint \frac{dz}{2\pi i} W(z) \tag{2.31}$$

over $\mathrm{W}(z)$ is manifestly conformally invariant, i.e. it commutes with all the Virasoro generators $[L_m, \mathbf{W}] = 0\,, \forall\, m$, and is invariant under conformal tranformations f: $f \circ \mathbf{W} = \mathbf{W}$. The latter implies in particular that we may make transformations of the form $f \circ \mathbf{W}\phi(0) = \mathbf{W}(f \circ \phi(0))$ inside correlation functions.

Using this one first notes that $\mathbf{W}$ is a derivation on the star algebra:

$$\mathbf{W}(A * B) = \mathbf{W}(A) * B + (-1)^{|W||A|} A * \mathbf{W}(B) \tag{2.32}$$

due to

$$\begin{aligned}
\langle \phi, \mathbf{W}(A * B) \rangle &= \langle \mathbf{W}^\star \phi, A * B \rangle \\
&= -\left\langle f_1 \circ \oint \frac{dz}{2\pi i} W(z)\, \phi(0)\ f_2 \circ A(0)\ f_3 \circ B(0) \right\rangle \\
&= \left\langle f_1 \circ \phi(0)\ f_2 \circ \oint \frac{dz}{2\pi i} W(z)\, A(0)\ f_3 \circ B(0) \right\rangle \\
&\quad + (-1)^{|A|} \left\langle f_1 \circ \phi(0)\ f_2 \circ A(0)\ \oint \frac{dz}{2\pi i} W(z)\, f_3 \circ B(0) \right\rangle \\
&= \left\langle \phi, (\mathbf{W}A) * B + (-1)^{|W||A|} A * (\mathbf{W}B) \right\rangle ,
\end{aligned} \tag{2.33}$$

where we have just performed some obvious contour deformations.

But a stronger statement is true, which was introduced in the context of purely cubic and vacuum string field theory [3]: CIT: Kishimo-toOhmori:2002

Let $C_{\rm L}$ be the left half of the unit circle and $C_{\rm R}$ be the right half and for any chiral W define

$$\mathbf{W}_{\rm L/R} := \int\limits_{C_{\rm L/R}} \frac{dz}{2\pi i} W(z)\;. \tag{2.34}$$

Using the derivation in equation (4.33) of [3] (see also the discussion in [11]) one shows that CIT: Kishimo-toOhmori:2002 CIT: Kluson:2003

$$(\mathbf{W}_{\rm R}A) * B \;=\; -(-1)^{|W||A|} A * (\mathbf{W}_{\rm L}B)\,. \tag{2.35}$$ LAB: left/right partial int

Note furthermore that for $\mathcal{I}$ the identity string field as discussed in [3], which satisfies CIT: Kishimo-toOhmori:2002

$$\mathcal{I} * A = A * \mathcal{I} = A \tag{2.36}$$ LAB: identity string field

on *most* of the string fields A, it follows in particular that

$$\mathbf{W}_{\rm L}(\mathcal{I}) \;=\; -\mathbf{W}_{\rm R}(\mathcal{I})\,. \tag{2.37}$$ LAB: left right annihilatio

This are the crucial identities used in [11] to express the graded star anticommutator by the action of an operator in the string's Hilbert space: CIT: Kluson:2003

Assume that the string field Φ can be written in the form

$$\Phi \;=\; \mathbf{W}_{\rm R}(\mathcal{I})\,, \tag{2.38}$$

From this we get the crucial relation between the graded star commutator and the action of operators in the string's Hilbert space:

$$[\Phi,\psi] \;=\; -\mathbf{W}\psi\,. \tag{2.39}$$ LAB: star commutator as

This follows by writing

$$\begin{aligned}
[\Phi,\psi] &= \Phi * \psi - (-1)^{|\Phi||\psi|}\psi * \Phi \\
&= (\mathbf{W}_{\rm R}\mathcal{I}) * \psi - (-1)^{|\Phi||\psi|}\psi * (\mathbf{W}_{\rm R}\mathcal{I}) \\
&\overset{(2.37)}{=} (\mathbf{W}_{\rm R}\mathcal{I}) * \psi + (-1)^{|\Phi||\psi|}\psi * (\mathbf{W}_{\rm L}\mathcal{I}) \\
&\overset{(2.35)}{=} -\left(\mathcal{I} * (\mathbf{W}_{\rm L}\psi) + (\mathbf{W}_{\rm R}\psi) * \mathcal{I}\right) \\
&\overset{(2.36)}{=} -\mathbf{W}\psi\,.
\end{aligned} \tag{2.40}$$

Hence any BRST operator (2.19) Q^{Φ_0} with $\Phi_0 = \mathbf{W}_R(\mathcal{I})$ can explicitly be written as

$$Q^{\Phi_0} \;=\; Q - \mathbf{W}\,. \tag{2.41}$$ LAB: deformed bosonic B

In particular, using $[Q_{\rm L},\mathbf{W}_{\rm R}] = 0$ one finds [11] that under a pure gauge transformation (2.29) has the expected form [6] CIT: Kluson:2003 CIT: Giannakis:2002

$$
\begin{aligned}
\Phi_0 &= e^{-\mathbf{W}_{\rm R}(\mathcal{I})} * (Q e^{\mathbf{W}_{\rm R}(\mathcal{I})}) \\
|\psi\rangle &\mapsto e^{-\mathbf{W}} |\psi\rangle \\
Q &\mapsto e^{-\mathbf{W}} \circ Q \circ e^{\mathbf{W}} \,.
\end{aligned}
\tag{2.42}
$$

LAB: certain gauge transf

Using this it is relatively easy to see [11] that the string field action expressed with respect to the new background describes a symmetry transformation on spacetime which manifests itself as a chiral boundary deformation of the worldsheet boundary CFT, as discussed in [13] (simple examples of such symmetry transformations are for instance translations in spacetime leading to translations of the respective D-branes,e .g. p. 32 of [13]).

CIT: Kluson:2003

CIT: RecknagelSchomerus:1999

CIT: RecknagelSchomerus:1999

The crucial step is the equation

$$
[\exp(-W_{\rm R}(\mathcal{I})) * Q * \exp(W_{\rm R}(\mathcal{I})), \Phi] = \exp(-W) \circ Q \circ \exp(W)(\Phi) \,,
\tag{2.43}
$$

LAB: star vs circ conjugat

which follows by using (2.39) as well as

$$
[A_{\rm L}, B_{\rm R}] = 0 \,.
\tag{2.44}
$$

LAB: left and right ops co

2.4.2 Deformed Virasoro generators

LAB: deformed Virasoro generators

Using (2.41) it is straightforward to deduce the deformation of the Virasoro generators as string field backgrounds are turned on.

Consider a string field of the form

$$
\begin{aligned}
\Phi &= \mathbf{W}_{\rm R}(\mathcal{I}) \\
\mathbf{W} &= \oint \frac{dz}{2\pi i} c(z)\, V(z)
\end{aligned}
\tag{2.45}
$$

where $V(z)$ is a chiral field of weight 2, so that $\mathbf{W}$ is the integral over a field of unit weight.

The string field equations of motion for this field translate into

$$
\begin{aligned}
0 &= Q\Phi + \Phi * \Phi \\
&= [Q, \mathbf{W}]_{\rm L}(\mathcal{I}) + \mathbf{W}_{\rm L}(\mathcal{I}) * \mathbf{W}_{\rm L}(\mathcal{I}) \\
&= [Q, \mathbf{W}]_{\rm L}(\mathcal{I}) + \frac{1}{2} \{\mathbf{W}_{\rm L}(\mathcal{I}), \mathbf{W}_{\rm L}(\mathcal{I})\} \\
&= [Q, \mathbf{W}]_{\rm L}(\mathcal{I}) + \frac{1}{2} (\mathbf{W} \circ \mathbf{W})_{\rm L}(\mathcal{I}) \,,
\end{aligned}
\tag{2.46}
$$

so that the star product equation in terms of $\mathbf{W}_{\rm L}(\mathcal{I})$ translates into an ordinary operator product equation in terms of $\mathbf{W}$:

$$
Q \cdot \mathbf{W} + \mathbf{W}^2 = 0 \,.
\tag{2.47}
$$

Furthermore, from (2.41) one finds that the deformed BRST operator reads

$$
\begin{aligned}
Q^{\Phi} &= Q + \mathbf{W} \\
&= \oint \frac{dz}{2\pi i} c(T^{\rm m} + V) + \frac{1}{2} c T^{\rm g} \,.
\end{aligned}
\tag{2.48}
$$

Therefore it corresponds to a deformation

$$
T \to T + V
\tag{2.49}
$$

of the Virasoro generators by a weight 2 field, which is in accord with the theory of canonical deformations (*cf.* [6, 14] and references given there).

CIT: Giannakis:2002,Schreiber:2004

2.5 Purely cubic/background free formulation

Equation (2.39) implies in particular that the action of the BRST operator can be written as a graded commutator

$$[Q_{\rm L}(\mathcal{I}),\psi] \stackrel{(2.39)}{=} Q(\psi)\ . \tag{2.50}$$

with the string field $Q_{\rm L}(\mathcal{I})$ which squares to zero:

$$\begin{aligned} Q_{\rm L}(\mathcal{I}) * Q_{\rm L}(\mathcal{I}) &= \frac{1}{2}\,[Q_{\rm L}(\mathcal{I}), Q_{\rm L}(\mathcal{I})] \\ &\stackrel{(2.39)}{=} [Q_{\rm L}, Q_{\rm L}](\psi) \\ &\stackrel{(2.44)}{=} Q^2(\psi) \\ &= 0\,. \end{aligned} \tag{2.51}$$

But this implies that $\Phi_0 = Q_{\rm L}(\mathcal{I})$ solves the equations of motion (2.6) for vanishing BRST operator. As observed in [2] this means that the action (2.4) is equivalent to the *background independent* action

CIT: Horowitz-LykkenRohmStrominger:1986

$$S_{\rm bf} := \frac{2}{3}\int \Phi * \Phi * \Phi\,. \tag{2.52}$$

2.6 Gauge transformations and chiral marginal deformations

LAB: Gauge transformations and chiral marginal deformations

[15]

CIT: Kluson:2002

2.7 Exact Solutions by the Lax pair method

The string field equations of motion

$$\begin{aligned} & Q\Phi + \Phi \star \Phi = 0 \\ \Leftrightarrow\ & (Q+\Phi)^2 = 0 \end{aligned} \tag{2.53}$$

are formally the 0-curvature condition of a covariant exterior derivative

$$\nabla = \mathbf{d} + A\,, \tag{2.54}$$

with $\mathbf{d} = Q$, $\Phi = A$ and $* = \wedge$.

Given any such connection it is well known that covariantly constant 0-forms in the fundamental representation exist if the connection is flat.

This follows from the fact that a covariantly constant element v by definition satisfies

$$\begin{aligned} & (\mathbf{d} + A)v = 0 \\ \Leftrightarrow\ & \partial_\mu v = -A_\mu v \end{aligned} \tag{2.55}$$

and that this implies the consistency condition:

$$\begin{aligned} 0 &= \partial_\mu \partial_\nu v - \partial_\nu \partial_\mu v \\ &= -\partial_\mu \left(A_\nu v\right) + \partial_\nu \left(A_\mu v\right) \\ &= \left(\partial_\nu A_\mu\right) v - \left(\partial_\mu A_\nu\right) v + A_\nu A_\mu v - A_\mu A_\nu v \\ &= ((\mathbf{d}+A)^2)_{\nu\mu} v\,. \end{aligned} \tag{2.56}$$

This method is used in [16] to find solutions to the equations of motion of string field theory by solving for the string field ψ in the linear equation

CIT: Lechtenfeld-PopovUhlmann:2002

$$Q\psi + \Phi * \psi = 0\,. \tag{2.57}$$

If solutions exist then Φ satisfies the string field equations of motion.

2.8 Calculational tools

2.8.1 Local oscillators

It was found in [17] that the OSFT vertices in operator formalism greatly simplify when certain 'unconventional' oscillators are used, which we shall call *local oscillators* here. CIT: KawanoOkuyama:2001

Recall the basic definition of [17]: CIT: KawanoOkuyama:2001

On the open string with Neuman boundary conditions and with spatial parameter $\sigma \in (0,\pi)$ the natural orthonormal set of functions is

$$\begin{aligned} \phi_0(\sigma) &= \frac{1}{\sqrt{\pi}} \\ \phi_n(\sigma) &= \sqrt{\frac{2}{\pi}}\cos n\sigma\,, \end{aligned} \tag{2.58}$$

which is complete in the sense that

$$\begin{aligned} \int_0^\pi d\sigma\,\phi_n(\sigma)\,\phi_m(\sigma) &= \delta_{n,m} \\ \sum_{n=0}^\infty \phi_n(\sigma)\;\phi_n(\sigma') &= \delta(\sigma,\sigma')\;. \end{aligned} \tag{2.59}$$

The ordinary worldsheet oscillators α_n^μ are the coefficients of the canonical fields $X^\mu(\sigma)$, $P_\mu(\sigma)$ with respect to this system:

$$\begin{aligned} X^\mu(\sigma) &= x^\mu + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\cos n\sigma \\ P_\mu(\sigma) &= \frac{1}{\pi}\left(p_\mu + \frac{1}{\sqrt{2\alpha'}}\sum_{n\neq 0}\eta_{\mu\nu}\alpha_n^\nu\cos n\sigma\right)\,, \end{aligned} \tag{2.60}$$

so that

$$\left[X^\mu(\sigma), P_\nu(\sigma')\right] = i\delta^\mu_\nu\,\delta(\sigma,\sigma') \Leftrightarrow \left\{ \begin{array}{l} [\alpha_n^\mu, \alpha_m^\nu] = \eta^{\mu\nu}\,n\,\delta_{n+m,0} \\ [x^\mu, p_\nu] = i\delta^\mu_\nu \end{array}\right. . \tag{2.61}$$

Following the notation of [17] let CIT: KawanoOkuyama:2001

$$\begin{aligned} b_n^\mu &:= \frac{1}{\sqrt{n}}\alpha_n^\mu \\ b_n^{\dagger\mu} &:= \frac{1}{\sqrt{n}}\alpha_{-n}^\mu\,, \qquad \forall\, m>0 \end{aligned} \tag{2.62}$$

LAB: usual creators/annih

be the associated creation/annihilation operators.

The crucial (though very simple) step now is the introduction of an alternative set of oscillators $a^\mu(\sigma)\,, a^{\dagger\mu}(\sigma)$, which satisfy the creation/annihilation algebra *locally*:

$$\left[a^\mu(\sigma), a^{\dagger\nu}(\sigma')\right] = \eta^{\mu\nu}\,\delta(\sigma,\sigma')\;. \tag{2.63}$$

These are obviously given by

$$a^\mu(\sigma) := \frac{1}{\sqrt{2}} \left(\sqrt{2\pi\alpha'} P^\mu(\sigma) - \frac{i}{\sqrt{2\pi\alpha'}} X^\mu(\sigma) \right) \tag{2.64}$$

or trivial rescalings thereof.

Their modes

$$a_n^\mu := \int \phi_n(\sigma)\ a^\mu(\sigma) \tag{2.65}$$

satisfy

$$\left[a_n^\mu, a_m^{\dagger\nu} \right] = \eta^{\mu\nu}\, \delta_{n,m} \tag{2.66}$$

and are related to the usual oscillators (2.62) by

$$\begin{aligned} a_0^\mu &= \sqrt{\alpha'} p^\mu - i \frac{1}{2\sqrt{\alpha'}} x^\mu \\ a_n^\mu &= \frac{1}{2} \left(\sqrt{n} + \frac{1}{\sqrt{n}} \right) b_n^\mu + \frac{1}{2} \left(\sqrt{n} - \frac{1}{\sqrt{n}} \right) b_n^{\dagger\mu}\,, \qquad \forall\, n > 0\,, \end{aligned} \tag{2.67}$$

which is nothing but a Bogoliubov transformation

$$\begin{aligned} a_n^\mu &= U b_n^\mu U^{-1} \\ &:= \cosh\theta_n b_n^\mu + \sinh\theta_n b_n^{\dagger\mu} \end{aligned} \tag{2.68}$$

with

$$\begin{aligned} U &= \exp\left(\frac{1}{2} \sum_{n=1}^{\infty} (\ln\sqrt{n})(b_n^2 - b_n^{\dagger 2}) \right) \\ &= \exp\left(\frac{1}{2} \sum_{n=1}^{\infty} (\ln\sqrt{n})(a_n^2 - a_n^{\dagger 2}) \right)\,. \end{aligned} \tag{2.69}$$

[...]

3. Superstring field theory

3.1 Heuristic derivation of the WZW-like SSFT action

This section reviews the motivation for the proposal of Berkovit's WZW-like NSSFT as discussed in [18]. CIT: Berkovits:2001

In the NS sector of the superstring the states in their natural picture are of the form

$$V = ce^{-\phi}\psi^\mu A_\mu(x) + \cdots . \tag{3.1}$$

The condition that V be part of the 'small' Hilbert space is

$$\eta_0 V = 0 , \tag{3.2}$$

i.e. that it does not depend on the ξ_0 mode.

This has picture -1. Problems with the picture raising operator in interaction terms suggest to use a field of 0 picture instead:

$$\Phi = \xi ce^{-\phi}\,\psi^\mu A_\mu + \cdots . \tag{3.3}$$

Now the task is to construct an action for Φ which reproduces the linearizes equations of motion for V and hand reasonable interaction terms. Since η_0 annihilates the ξ_0 mode again

$$\eta_0\Phi = V \tag{3.4}$$

we are looking for linearized equations of motion

$$Q\eta_0\Phi = 0 . \tag{3.5}$$

Noting that Q and η_0 are both nilpotent and mutually anticommuting

$$Q^2 = \eta_0^2 = \{Q, \eta_0\} = 0 \tag{3.6}$$

this is reminiscent of an equation of the form

$$\bar\partial\partial\Phi = 0 \tag{3.7}$$

for Grassmannian partial derivatives.

The corresponding action is obviously

$$S_{\text{lin}} \propto \langle \Phi Q\eta_0\Phi\rangle . \tag{3.8}$$

LAB: linear NSSFT actior

It has the gauge invariance

$$\delta\Phi = \eta_0\tilde\Lambda + Q\Lambda . \tag{3.9}$$

Due to $\{\eta_0, \xi_0\} = 1$, $\eta_0^2 = 0 = \xi_0^2$ every state has a component in the η_0 vacuum (small Hilbert space) and one in the ξ_0 excitation of it. This means that the first parameter $\tilde\Lambda$

of the above gauge transformation can be used to eliminate the η_0-vacuum component so that we can always obtain Φ in the form

$$\Phi = \xi_0 V \tag{3.10}$$

for V in the small Hilbert space, as required for physical states.

Furthermore, the other gauge parameter, Λ, obviously yields the ordinary BRST-cohomology gauge symmetry: Within the above gauge we have to use $\Lambda = -\xi_0\Omega$ which then corresponds to the usual linearized gauge transformation

$$\delta V = Q\Omega\,. \tag{3.11}$$

For these reasons the action (3.8) is reasonable and a non-linear generalization with the above gauge invariances is sought for. It turns out the the WZW model has the right properties. This is discussed in the next section:

3.2 The WZW model

The version of superstring field theory due to Berkovits is based on the form of the Wess-Zumino-Witten (WZW) action (e.g. [19] §15.1.2), which is the action for a group-valued field $g(x)$ on a 3-dimensional manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ given by CIT: DiFrancescoMathieuSenechal:1997

$$\begin{aligned} S_{\mathrm{WZW}} &=:= S_{\mathrm{kin}} + S_{\mathrm{top}} \\ &:= \frac{1}{2}\mathrm{Tr}\int\limits_{\partial\mathcal{M}} (g^{-1}\mathbf{d}g)\wedge\star(g^{-1}\mathbf{d}g) + \frac{1}{3}\mathrm{Tr}\int\limits_{\mathcal{M}} (g^{-1}\mathbf{d}g)\wedge(g^{-1}\mathbf{d}g)\wedge(g^{-1}\mathbf{d}g)\,. \end{aligned} \tag{3.12}$$

LAB: WZW action

We take the metric on ∂M to be *pseudo Riemannian* of signature $(-,+)$.

The variation[3] yields the classical equations of motion

$$\delta S_{\mathrm{WZW}} = 0 \;\Leftrightarrow\; \mathbf{d} \star g^{-1}\mathbf{d}g - (g^{-1}\mathbf{d}g) \wedge (g^{-1}\mathbf{d}g) = 0\,, \tag{3.15}$$ LAB: WZW EOM

which applies at the boundary $\partial\mathcal{M}$.

This can be rewritten as

$$\mathbf{d}A + A \wedge A = 0 \tag{3.16}$$

by setting

$$A := -\star g^{-1}\mathbf{d}g\,. \tag{3.17}$$ LAB: CS like WZW forms

This way a certain formal analogy to the equations of motion of Chern-Simons theory (2.6) is exhibited and in particular the gauge invariance

$$\delta A = \mathbf{d}\Lambda + [A, \Lambda] \tag{3.18}$$ LAB: CS gauge in WZW

of the classical equations of motion is immediate, which of course comes from the linear order of the finite gauge transformation

$$\mathbf{d} + A\wedge \to e^{-\Lambda} \circ (\mathbf{d} + A\wedge) \circ e^{\Lambda} \tag{3.19}$$

of the gauge covariant derivative. But furthermore the equations of motion are invariant under

$$\delta A = g^{-1}(\star\mathbf{d}\Lambda)g \tag{3.20}$$ LAB: general left gauge tr

[3]

$$\begin{aligned}
\delta S_{\mathrm{kin}} &= \mathrm{Tr}\int\limits_{\partial\mathcal{M}} (\delta g^{-1}\mathbf{d}g) \wedge \star(g^{-1}\mathbf{d}g) + \mathrm{Tr}\int\limits_{\partial\mathcal{M}} (g^{-1}\mathbf{d}\delta g) \wedge \star(g^{-1}\mathbf{d}g) \\
&= \mathrm{Tr}\int\limits_{\partial\mathcal{M}} (\delta g^{-1}\mathbf{d}g) \wedge \star(g^{-1}\mathbf{d}g) - \mathrm{Tr}\int\limits_{\partial\mathcal{M}} (\mathbf{d}g^{-1})\delta g \wedge \star(g^{-1}\mathbf{d}g) - \mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g \wedge \mathbf{d}\star(g^{-1}\mathbf{d}g) \\
&= -\mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g g^{-1}(\mathbf{d}g) \wedge \star(g^{-1}\mathbf{d}g) - \mathrm{Tr}\int\limits_{\partial\mathcal{M}} (g^{-1}\mathbf{d}g) \wedge \star(\mathbf{d}g^{-1})\delta g - \mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g \wedge \mathbf{d}\star(g^{-1}\mathbf{d}g) \\
&= -\mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g(g^{-1}\mathbf{d}g) \wedge \star(g^{-1}\mathbf{d}g) - \mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g(g^{-1}\mathbf{d}g) \wedge \star(\mathbf{d}g^{-1})g - \mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g \wedge \mathbf{d}\star(g^{-1}\mathbf{d}g) \\
&= -\mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g\, \mathbf{d}\star(g^{-1}\mathbf{d}g)
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\delta S_{\mathrm{top}} &= \mathrm{Tr}\int\limits_{\mathcal{M}} (\delta g^{-1}\mathbf{d}g) \wedge (\delta g^{-1}\mathbf{d}g)^2 + \mathrm{Tr}\int\limits_{\mathcal{M}} (g^{-1}\mathbf{d}\delta g) \wedge (\delta g^{-1}\mathbf{d}g)^2 \\
&= -\mathrm{Tr}\int\limits_{\mathcal{M}} g^{-1}\delta g(g^{-1}\mathbf{d}g) \wedge (\delta g^{-1}\mathbf{d}g)^2 - \mathrm{Tr}\int\limits_{\mathcal{M}} \delta g\,\mathbf{d}\;\;(\delta g^{-1}\mathbf{d}g)^2 g^{-1} \;\; + \mathrm{Tr}\int\limits_{\partial\mathcal{M}} \delta g(g^{-1}\mathbf{d}g)^2 g^{-1} \\
&= \mathrm{Tr}\int\limits_{\partial\mathcal{M}} g^{-1}\delta g\,(g^{-1}\mathbf{d}g) \wedge (g^{-1}\mathbf{d}g)
\end{aligned} \tag{3.14}$$

if $\star\mathbf{d}\Lambda = -\mathbf{d}\Lambda$. This will be related to the usual form of the gauge transformation of the WZW model below in (3.28).

But due to the special nature of A in (3.17) there is a further gauge invariance of the equation (3.15), namely

$$\delta A = g^{-1}(\mathbf{d}\Lambda)g \tag{3.21}$$

Since the action is conformally invariant we can find coordinates $z, \bar{z}$ such that

$$\begin{aligned} \star\, dz &= -\, dz \\ \star\, d\bar{z} &= \, d\bar{z}\,. \end{aligned} \tag{3.22}$$

LAB: WZW Hodge

In these coordinates the equation of motion (3.15) is equivalent to

$$\begin{aligned} 0 &= \partial(g^{-1}\bar{\partial}g) + \bar{\partial}(g^{-1}\bar{\partial}g) - \big((g^{-1}\partial g)(g^{-1}\bar{\partial}g) - (g^{-1}\bar{\partial}g)(g^{-1}\partial g)\big) \\ &= \partial(g^{-1}\bar{\partial}g) + \bar{\partial}(g^{-1}\bar{\partial}g) + \big(\partial(g^{-1}\bar{\partial}g) - \bar{\partial}(g^{-1}\partial g)\big) \end{aligned} \tag{3.23}$$

LAB: WZW eom in comp

and hence to

$$\partial(g^{-1}\bar{\partial}g) = 0\,. \tag{3.24}$$

LAB: WZW eom

The general solution to the equations of motion (3.24) is

$$g(z,\bar{z}) = l(z)\, r(\bar{z}) \tag{3.25}$$

for arbitrary holomorphic l and antiholomorphic r.

In complex coordinates the variation of the action reads

$$\delta S_{\mathrm{WZW}} \propto \mathrm{Tr} \int\limits_{\partial M} d^2z\; g^{-1}\delta g \partial(g^{-1}\bar{\partial}g)\,, \tag{3.26}$$

which can be seen to vanish for the general *gauge transformation*

$$\delta g(z,\bar{z}) = \alpha(z)\, g(z,\bar{z}) + g(z,\bar{z})\, \beta(\bar{z}) \tag{3.27}$$

LAB: WZW gauge trafo

for arbitrary holomorphic α and antiholomorphic β.

This gauge transformation acts on the Lie-algebra 1-forms (3.17) as

$$\begin{aligned} b^{-1}g^{-1}a^{-1} \star \mathbf{d}(agb) &\approx \left(g^{-1} - \beta g^{-1} - g^{-1}\alpha\right) \star \mathbf{d}\,(g + \alpha g + g\beta) \\ &\approx g^{-1} \star \mathbf{d}g - \beta g^{-1} \star \mathbf{d}g - g^{-1}\alpha \star \mathbf{d}g + g^{-1} \star (\mathbf{d}\alpha)g + g^{-1}\alpha \star \mathbf{d}g + g^{-1} \star (\mathbf{d}g)\beta + \star\mathbf{d}\beta \\ &= g^{-1} \star \mathbf{d}g + \left[g^{-1} \star \mathbf{d}g, \beta\right] + \star\mathbf{d}\beta + g^{-1} \star (\mathbf{d}\alpha)g\,. \end{aligned} \tag{3.28}$$

LAB: '

By comparison with (3.18) and (3.20) one finds that β induces a gauge transformation iff $\star\mathbf{d}\beta = \mathbf{d}\beta$ and α induces a gauge transformation iff $\star\mathbf{d}\alpha = -\mathbf{d}\alpha$, which is, by (3.22), equivalent to α being holomorphic and to β being antiholomorphic, in accord with (3.27).

The same equations of motion are obtained (*cf.* [20]) when $z, \bar{z}$ are *anticommuting* CIP: Kluson:2001 coordinates so that

$$\begin{aligned} &\partial^2 = 0 = \bar\partial^2 \\ &\partial\bar\partial = -\bar\partial\partial \\ &dz \wedge d\bar z = d\bar z \wedge dz\,. \end{aligned} \tag{3.29}$$

This simply changes two signs in (3.23)

$$\begin{aligned} 0 &= \partial(g^{-1}\bar\partial g) - \bar\partial(g^{-1}\bar\partial g) - \big((g^{-1}\partial g)(g^{-1}\bar\partial g) + (g^{-1}\bar\partial g)(g^{-1}\partial g)\big) \\ &= \partial(g^{-1}\bar\partial g) - \bar\partial(g^{-1}\bar\partial g) + \big(\partial(g^{-1}\bar\partial g) + \bar\partial(g^{-1}\partial g)\big) \end{aligned} \tag{3.30}$$ LAB: WZW eom in comp

and still leads to

$$\partial(g^{-1}\bar\partial g) = 0\,. \tag{3.31}$$ LAB: final WZW eom

Berkovits' superstring field theory in the NS sector (NSFT) is obtained from these general considerations by identifying the anticommuting ∂ and $\bar\partial$ with the ghost η_0 and the BRST operator Q, respectively.

As noted in [20], classical solutions of the WZW model give perturbations of one of the partial derivatives in much the same way as for Chern-Simons theory: CIT: Kluson:2001

Let g_0 be a solution to (3.31) and consider which equation of motion has to be satisfied by the perturbation

$$g = g_0 h \tag{3.32}$$ LAB: WZW perturbation

about this solution:

$$\begin{aligned} 0 &= \partial\left(h^{-1}g_0^{-1}\bar\partial(g_0 h)\right) \\ &= \partial\left(h^{-1}\bar\partial h\right) + \partial\left(h^{-1}g_0^{-1}(\bar\partial g_0)h\right) \\ &\overset{(3.31)}{=} \partial\left(h^{-1}\bar\partial h\right) + \partial\left(h^{-1}g_0^{-1}(\bar\partial g_0)h\right) - \partial\left(h^{-1}h g_0^{-1}(\bar\partial g_0)\right) \\ &= \partial\left(h^{-1}\left(\bar\partial h + [g_0^{-1}\bar\partial g_0, h]\right)\right)\,. \end{aligned} \tag{3.33}$$ LAB: shifted WZW eom

When we promote the group elements g to multiplication operators $\hat g$ on some Hilbert space so that $\left[\bar\partial, \hat g\right] = \widehat{\bar\partial g}$ then the ordinary equation of motion (3.31) can be written as

$$\left[\partial, \hat g^{-1}\left[\bar\partial, \hat g\right]\right] = 0 \tag{3.34}$$

and the shifted equation (3.33) as

$$\left[\partial, \hat h^{-1}\left[\bar\partial^{g_0}, \hat h\right]\right] = 0\,, \tag{3.35}$$

where the deformed antiholomorphic derivative

$$\bar\partial^{g_0} := \hat g_0^{-1} \circ \bar\partial \circ \hat g_0 \tag{3.36}$$ LAB: deformed partial WZ

has been used.

Obviously a shift h of a classical solution g_0 as in (3.32) leaves $\bar\partial$ invariant iff $\bar\partial g_0 = 0$, i.e. if g_0 is purely holomorphic, in which case it is pure gauge, in accord with (3.27).

It is easy to see that $\bar\partial^{g_0}$ still has the desired properties

$$\begin{aligned}
&(\bar\partial^{g_0})^2 = 0\\
&\int \bar\partial^{g_0}(\cdots) = 0\\
&\left\{\bar\partial^{g_0}, \partial\right\} = 0\,.
\end{aligned} \tag{3.37}$$

As has been noted in [11], while in the bosonic SFT nilpotence of the deformed BRST operator depends on the equations of motion for the background field, here it is instead the anticommutation of ∂ with $\bar\partial^{g_0}$ which requires the equations of motion, while $(\bar\partial^{g_0})^2$ is automatic. CIT: Kluson:2003

On the other hand, it follows that the nilpotence $(\mathbf{d}^{g_0})^2$ of the deformed total exterior derivative (on the space of Grassmann coordinates $z, \bar z$!)

$$\mathbf{d}^{g_0} := dz \wedge \partial + d\bar z \wedge \bar\partial^{g_0} \tag{3.38}$$

depends on the equations of motion.

3.3 Background free formulation

As in the bosonic theory, one can find a *background-free version* of Berkovits' string field theory by constructing a suitable string field which emulates the BRST operator. This has been demonstrated in [21]. It is based on the observation of [22] that the superstring BRST operator can be written as CIT: Kluson:2001b CIT: AcostaBerkovitsChandia:1999

$$Q = \exp(-R) \circ Q_0 \circ \exp(R) \tag{3.39}$$

LAB: susy BRST as conju

where

$$Q_0 = \oint \frac{dz}{2\pi i}\gamma^2 b \tag{3.40}$$

is a background-free pure-ghost BRST operator and

$$R = \oint \frac{dz}{2\pi i}\left(cG_m e^{-\phi}e^{\chi} - \frac{1}{4}\partial(e^{-2\phi})e^{2\chi}c\partial c\right) \tag{3.41}$$

LAB: superstring R

contains the information about the background, encoded in the worldsheet matter supercurrent G_m.

Let $\bar\partial^{(0)} = Q_0$ be the 'background free' antiholomorphic partial derivative in the WZW action. The question is then:

Can we find a solution g_0 of the associated equations of motion

$$\partial(g^{-1}\bar\partial^{(0)}g) = 0 \tag{3.42}$$

such that the deformed partial derivative (3.36) for the above context, is just the ordinary BRST operator $\bar\partial = Q$:

$$\widehat{\bar\partial q} \stackrel{!}{=} \left[g_0^{-1}\circ\bar\partial^{(0)}\circ g, \hat q\right]\,. \tag{3.43}$$

LAB: bf BRST as commu

Using the standard construction introduced in [2] and reviewed in §2.4 (p.8) one obviously has to set

CIT: Horowitz-LykkenRohmStrominger:1986

$$g_0 = \exp(R_{\rm L}(\mathcal{I})) \tag{3.44}$$

LAB: SSFT bf background

with R from (3.41). That's because

$$\begin{aligned}
\left[\exp(-R_{\rm L}(\mathcal{I})) \circ \bar\partial^{(0)} \circ \exp(R_{\rm L}(\mathcal{I})), \hat q\right] &= \left[\bar\partial^{(0)}, \hat q\right] + \left[\exp(-R_{\rm L}(\mathcal{I}))\left(\bar\partial^{(0)} \exp(R_{\rm L}(\mathcal{I}))\right), \hat q\right] \\
&\overset{(2.39)}{=} \widehat{\bar\partial^{(0)} q} + \left(\exp(-R)\left[\bar\partial^{(0)}, \exp(R)\right]\right)(q)\hat{} \\
&\overset{(3.39)}{=} \widehat{\bar\partial q}\,.
\end{aligned} \tag{3.45}$$

LAB: partial from R

The only thing that one has to check is if g_0 satisfies its equations of motion. This follows as

$$\begin{aligned}
\partial\left(g_0^{-1}\bar\partial^{(0)} g_0\right) &\overset{(3.45)}{=} \partial\left(\left(\bar\partial - \bar\partial^{(0)}\right)_{\rm L}(\mathcal{I})\right) \\
&= -\left(\bar\partial - \bar\partial^{(0)}\right)_{\rm L}(\partial\mathcal{I}) \\
&= 0\,.
\end{aligned} \tag{3.46}$$

This insight of [21] has later been generalized from NSFT to RNS-SSFT in [23].

CIT: Kluson:2001b
CIT: Sakaguchi:2001

3.4 CFTs from superstring field backgrounds and deformed super Virasoro generators

LAB: CFTs from superstring field backgrounds and deformed super Virasoro generators

The above factorization (3.43) (3.44) of the BRST operator facilitates the analysis of deformations of the matter SCFT as the background is perturbed.

Consider a chiral field $a(z)$ from the matter sector of the worldsheet theory and of conformal weight $w(a) = 3/2$. From this we can construct the string field

$$A := \left(\int_{C_{\rm L}} \frac{dz}{2\pi i} c\xi e^{-\phi} a(z)\right)(\mathcal{I})\,. \tag{3.47}$$

When $a(z)$ can be found such that this solves the SSFT equations of motion we find a deformed BRST operator

$$\tilde Q(\psi) = \left[e^{-A} e^{-R_{\rm L}(\mathcal{I})} \circ Q \circ e^{R_{\rm L}(\mathcal{I})} e^{A}, \psi\right]\,. \tag{3.48}$$

But R and A commute, due to the c-ghost present in both. This means that

$$\tilde Q(\psi) = \left[e^{-(R_{\rm L}(\mathcal{I})+A)} \circ Q \circ e^{(R_{\rm L}(\mathcal{I})+A)}, \psi\right]\,. \tag{3.49}$$

By comparison with (3.41) it follows that the background shift induces the deformation

$$G_{\rm m} \to G_{\rm m} + a\,, \tag{3.50}$$

i.e. the supercurrent is deformed by a weight 3/2 field. This is in accord with the theory of canonical deformation of the closed string (*cf.* [14] and references given there) when appropriately adapted for the open string:

CIT: Schreiber:2004

4. Deformation theory

4.1 Deformations and backgrounds using the Morse theory technique

LAB: Deformations and backgrounds using the Morse theory technique

We can map the open string correlators $\langle \cdots \rangle_{\mathrm{UHP}}$ which enter the OSFT action to closed string correlators $\langle \cdots \alpha \rangle_{\mathrm{plane}}$ by using *boundary states* $|\alpha\rangle$ (reviewed in §C.2 (p.46)) on the comlex plane, which enforce the appropriate boundary condition on the unit circle.

As discussed above, a shift in the string field corresponds to a deformation of the BRST operator, while the correlator is kept fix.[4] In terms of the boundary state formalism this means that the correlators are always to be evaluated with respect to a fixed given boundary state $|\alpha\rangle$. On the other hand the deformation of the BRST operator translates to a deformation of the Virasoro generators $L_n \mapsto L_n^{(\Phi)}$, $\bar{L}_n \mapsto \bar{L}_n^{(\Phi)}$.

But the boundary state mechanism always strictly requires the constraint (C.12) on the boundary state which is in general true for the deformed Virasoro generators and the fixed boundary state $|\alpha\rangle$ only if

$$L_n^{(\Phi)} - \bar{L}_{-n}^{(\Phi)} = L_n - \bar{L}_{-n} \,. \tag{4.1}$$

LAB: rep gen independent

As discussed in [14] this condition also follows from the canonical analysis of the (closed) string, where its physical interpretation is transparent: It simply says that the generator of worldsheet-spatial reparameterizations of the string, being a Lie derivative on loop space, is independent of the background in which the string propagates.

CIT: Schreiber:2004

For the superstring the condition (4.1) already greatly restricts the form of allowed deformations of the supercurrents:

Let d_r and $d_r^\dagger$ be the modes of the polar combinations of the left- and right-moving suppercurrents

$$\begin{aligned} d_r &:= G_r + i\bar{G}_{-r} \\ d_r^\dagger := (d_r)^\dagger &= G_r - i\bar{G}_{-r} \,, \end{aligned} \tag{4.2}$$

LAB: polar supercurrent r

which are the 'square roots' of the reparameterization generator

$$\mathcal{L}_n := -i\left(L_n - \bar{L}_{-n}\right) \,, \tag{4.3}$$

LAB: rep gen

i.e.

$$\{d_r, d_s\} = \{d_r^\dagger, d_s^\dagger\} = 2i\mathcal{L}_{r+s} \,. \tag{4.4}$$

Under a deformation the right hand side of this equation must stay invariant (4.1) so that

$$\begin{aligned} d_r^\Phi &:= d_r + \Delta_\Phi d_r \\ d_r^{\dagger\Phi} &:= d_r^\dagger + (\Delta_\Phi d_r)^\dagger \end{aligned} \tag{4.5}$$

[4] As discussed in §2.6 (p.11) this can (at least in special cases) equivalently be translated by a field redefinition to a setup where the BRST operator remains unaffected and only the correlator is deformed (by a boundary deformation), but here we concentrate on the situation in terms of the BRST deformation.

implies that the shift $\Delta_\Phi d_r$ of d_r has to satisfy

$$\{d_r, \Delta_\Phi d_s\} + \{d_s, \Delta_\Phi d_r\} + \{\Delta_\Phi d_r, \Delta_\Phi d_s\} = 0\,. \tag{4.6}$$ LAB: shift in loop space e

One large class of solutions of this equation is

$$\Delta_\Phi d_r = A^{-1}\,[d_r, A]\;, \qquad \text{for } [\mathcal{L}_n, A] = 0\ \forall\, n\,, \tag{4.7}$$ LAB: similarity transform

where A is any even graded operator that is spatially reparameterization invariant, i.e. which commutes with (4.3).

When this is rewritten as

$$\begin{aligned} d_r^\Phi &= A^{-1} \circ d_r \circ A \\ d_r^{\dagger\Phi} &= A^\dagger \circ d_r^\dagger \circ A^{\dagger -1} \end{aligned} \tag{4.8}$$ LAB: simtrafo on susy cur

one sees explicitly that the formal structure involved here is a direct generalization of that used in [24] in the study of the relation of deformed generators in supersymmetric quantum *mechanics* to Morse theory. Here we are are concerned with the direct generalization of this mechanism from $1+0$ to $1+1$ dimensional supersymmetric field theory. CIT: Witten:1982

In $1+0$ dimensional SQFT (i.e. supersymmetric quantum mechanics) relation (4.8) is sufficient for the deformation to be truly an isomorphism of the algebra of generators. In 1+1 dimensions, on the superstring's worldsheet there is however one further necessary condition for this to be the case. Namely the (modes of the) new worldsheet Hamiltonian constraint $H_n = L_n + \bar{L}_- n$ must clearly be defined as

$$H_n^\Phi := \frac{1}{2}\left\{d_r^\Phi, d_{n-r}^{\dagger\Phi}\right\} - \delta_{n,0}\frac{c}{12}\left(4r^2 - 1\right) \tag{4.9}$$ LAB: def deformed worlds

and (4.6) alone does not guarantee that this is *unique* for all $r \neq n/2$. *If* it is, however, then the Jacobi identity already implies that

$$\begin{aligned} G_r^\Phi &:= \frac{1}{2}\left(d_r^\Phi + d_r^{\dagger\Phi}\right) \\ L_n^\Phi &:= \frac{1}{4}\left(\left\{d_r^\Phi, d_{n-r}^{\dagger\Phi}\right\} + \left\{d_r^\Phi, d_{n-r}^\Phi\right\}\right) - \delta_{r,n/2}\frac{c}{24}\left(4r^2-1\right) \\ \bar{G}_r^\Phi &:= -\frac{i}{2}\left(d_{-r}^\Phi - d_{-r}^{\dagger\Phi}\right) \\ L_n^\Phi &:= \frac{1}{2}\left(\left\{d_{-r}^\Phi, d_{r-n}^{\dagger\Phi}\right\} - \left\{d_{-r}^\Phi, d_{r-n}^\Phi\right\}\right)\,, \qquad \forall\, r \neq n/2 \end{aligned} \tag{4.10}$$

generate two mutually commuting copies of the super Virasoro algebra.

In order to see this first note that the two copies of the unperturbed Virasoro algebra in terms of the 'polar' generators $d_r, d_r^\dagger, i\mathcal{L}_m, H_m$ read

$$\begin{aligned} \{d_r, d_s\} &= 2\,i\mathcal{L}_{r+s} = \left\{d_r^\dagger, d_s^\dagger\right\} \\ [i\mathcal{L}_m, d_r] &= \frac{m-2r}{2} d_{m+r} \\ \left[i\mathcal{L}_m, d_r^\dagger\right] &= \frac{m-2r}{2} d_{m+r}^\dagger \end{aligned}$$

$$\begin{aligned}
&[i\mathcal{L}_m, i\mathcal{L}_n] = (m-n)i\mathcal{L}_{m+n} \\
&[i\mathcal{L}_m, H_n] = (m-n)i\mathcal{H}_{m+n} + \frac{c}{6}(m^3-m)\delta_{m,-n} \\
&[H_m, d_r] = \frac{m-2r}{2} d^\dagger_{m+r} \\
&\left[H_m, d^\dagger_r\right] = \frac{m-2r}{2} d_{m+r} \\
&[H_m, H_n] = (m-n)\, i\mathcal{L}_{m+n}\, .
\end{aligned} \tag{4.11}$$

Now check that these relations are obeyed also by the deformed generators $d^\Phi_r, d^{\dagger\Phi}_r, i\mathcal{L}_m, H^\Phi_m$ using the two conditions (4.8) and (4.9):

First of all the relations

$$\begin{aligned}
\left[i\mathcal{L}_m, d^\Phi_r\right] &= \frac{m-2r}{2} d^\Phi_{m+r} \\
\left[i\mathcal{L}_m, d^{\dagger\Phi}_r\right] &= \frac{m-2r}{2} d^{\dagger\Phi}_{m+r}
\end{aligned} \tag{4.12}$$

LAB: bracket Lm dr

follow simply from (4.8) and the original bracket $[L_m, G_r] = \frac{m-2r}{2} G_{m+r}$ and immediately imply

$$[i\mathcal{L}_m, i\mathcal{L}_n] = (m-n)i\mathcal{L}_{m+n} \tag{4.13}$$

(note that here the anomaly of the left-moving sector cancels that of the right-moving one).

Furthermore

$$\begin{aligned}
\left[i\mathcal{L}_m, H^\Phi_n\right] &= \left[i\mathcal{L}_m, \frac{1}{2}\left\{d^\Phi_r, d^{\dagger\Phi}_{n-r}\right\}\right] \\
&\overset{(4.12)}{=} \frac{m-2r}{4}\left\{d^\Phi_{m+r}, d^{\dagger\Phi}_{n-r}\right\} + \frac{m-2(n-r)}{4}\left\{d^\Phi_r, d^{\dagger\Phi}_{m+n-r}\right\} \\
&\overset{(4.9)}{=} (m-n)H^\Phi_{m+n} + \delta_{m,-n}\frac{c}{6}\left(\frac{m-2r}{4}(4(m+r)^2-1) + \frac{m-2(n-r)}{4}(4r^2-1)\right) \\
&= (m-n)H^\Phi_{m+n} + \delta_{m,-n}\frac{c}{6}\left(m^3-m\right)\, .
\end{aligned} \tag{4.14}$$

(Here the anomalies from both sectors add.)

The commutator of the Hamiltonian with the supercurrents is obtained for instance by first writing:

$$\begin{aligned}
\left[H^\Phi_m, d^\Phi_r\right] &= \frac{1}{2}\left[\left\{d^\Phi_r, d^{\dagger\Phi}_{m-r}\right\}, d^\Phi_r\right] \\
&= -\frac{1}{2}\left[\left\{d^\Phi_r, d^\Phi_r\right\}, d^{\dagger\Phi}_{m-r}\right] - \frac{1}{2}\left[\left\{d^\Phi_r, d^{\dagger\Phi}_{m-r}\right\}, d^\Phi_r\right] \\
&= -\left[i\mathcal{L}_{2r}, d^{\dagger\Phi}_{m-r}\right] - \left[H^\Phi_m, d^\Phi_r\right] \\
&= (m-2r)d^{\dagger\Phi}_{m+r} - \left[H^\Phi_m, d^\Phi_r\right]\, ,
\end{aligned} \tag{4.15}$$

from which it follows that

$$\left[H^\Phi_m, d^\Phi_r\right] = \frac{(m-2r)}{2} d^{\dagger\Phi}_{m+r} \tag{4.16}$$

and similarly

$$\left[H_m^\Phi, d_r^{\dagger\Phi}\right] = \frac{(m-2r)}{2} d_{m+r}^\Phi \,. \tag{4.17}$$

This can finally be used to obtain

$$\left[H_m^\Phi, H_n^\Phi\right] = (m-n) i\mathcal{L}_{m+n} \,. \tag{4.18}$$

In summary this shows that every operator A which

1. commutes with $i\mathcal{L}_m$
2. is such that $\left\{A^{-1} d_r A, A^\dagger d_{n-r}^\dagger A^{\dagger -1}\right\} - \delta_{n,0}\frac{c}{12}(4r^2-1)$ is *independent* of r

defines a consistent deformation of the super Virasoro generators and hence a string background which satisfies the classical equations of motion of string field theory.

In [14] it was shown how at least all massless NS and NS-NS backgrounds of the closed string can be obtained by deformations A of the form $A = e^{\mathbf{W}}$, where $\mathbf{W}$ is related to the vertex operator of the respective background field. CIT: Schreiber:2004

In the special case where A is *unitary* the similarity transformations (4.8) of d and $d^\dagger$ and hence of all other elements of the super-Virasoro algebra are identical and the deformation is nothing but a unitary transformation. It was discussed in [14] that gauge transformations of the background fields, such as reparameterizations or gauge shifts $B \mapsto B + dA$ are described by such unitary transformation, precisely as one would expect from string field theory considerations (*cf.* §2.3 (p.5)) CIT: Schreiber:2004

In particular, a gauge field background was shown to be induced by the Wilson line of the gauge field along the closed string.

By using boundary state formalism (see §C.2 (p.46) for a brief review) these facts nicely generalize to the *open* string. When the closed string is 'cut open' we can apply the respective unitary transformation to the boundary state. From a boundary state $|\alpha\rangle$ describing a bare brane we should hence get the boundary state of that brane with a gauge field turned on by writing $U|\alpha\rangle$, where U is the Wilson line of that gauge field along the closed string at the boundary.

Indeed, this was shown to be the correct boundary state in [25, 26, 27]. CIT: MaedaNakatsuOonishi:2004,Hashimoto:2000

4.2 Boundary states and loop space formalism

The boundary state describing the space-filling D9-brane in Minkowski space is, according to (C.11), given by the constraints

$$\begin{aligned} \left(\alpha_n^\mu + \bar\alpha_{-n}^\mu\right)|\alpha\rangle &= 0\,, \qquad \forall\, n,\mu \\ \left(\psi_r^\mu - i\bar\psi_{-r}^\mu\right)|\alpha\rangle &= 0\,, \qquad \forall\, r,\mu\,, \end{aligned} \tag{4.19}$$

LAB: boundary constraint

(in the open string R sector).

As disucssed in [14] we can think of the super-Virasoro constraints as a Dirac-Kähler CIT: Schreiber:2004 system on the exterior bundle over loop space $\mathcal{L}(\mathcal{M})$ with coordinates

$$X^{(\mu,\sigma)} = \frac{1}{\sqrt{2\pi}} X_0^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu - \tilde{\alpha}_{-n}^\mu\right) e^{in\sigma} , \tag{4.20}$$

holonomic vector fields

$$\partial_{(\mu,\sigma)} = i\sqrt{\frac{T}{4\pi}} \sum_{n=-\infty}^{\infty} \left(\alpha_n^\mu + \tilde{\alpha}_{-n}^\mu\right) e^{in\sigma} \tag{4.21}$$

differential form creators

$$\begin{aligned} \mathcal{E}^{\dagger(\mu,\sigma)} &= \frac{1}{2}\left(\psi_+^\mu(\sigma) + i\psi_-^\mu(\sigma)\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_r \left(\bar{\psi}_{-r} + i\psi_r\right) e^{ir\sigma} \end{aligned} \tag{4.22}$$

LAB: loop space differenti

and annihilators

$$\begin{aligned} \mathcal{E}^{(\mu,\sigma)} &= \frac{1}{2}\left(\psi_+^\mu(\sigma) - i\psi_-^\mu(\sigma)\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_r \left(\bar{\psi}_{-r} - i\psi_r\right) e^{ir\sigma} . \end{aligned} \tag{4.23}$$

In the polar form (4.2) the fermionic super Virasoro constraints are identified with the exterior derivative on loop space

$$\mathbf{d}_K(\sigma) = \mathcal{E}^{\dagger\mu}\partial_\mu(\sigma) + iTX'^\mu\mathcal{E}_\mu(\sigma) , \tag{4.24}$$

LAB: deformed extd on lo

deformed by the reparameterization Killing vector

$$K^{(\mu,\sigma)} := TX'^\mu(\sigma) , \tag{4.25}$$

as well as its adjoint.

Using this formulation of the super-Virasoro constraints it would seem natural to represent them on a Hilbert space whose 'vacuum' state $|\mathrm{vac}\rangle$ is the *constant 0-form* on loop space, i.e.

$$\partial_{(\mu,\sigma)}\,|\mathrm{vac}\rangle = 0 = \mathcal{E}_{(\mu,\sigma)}\,|\mathrm{vac}\rangle \qquad \forall\,\mu,\sigma\,. \tag{4.26}$$

LAB: condition on consta

While this is not the usual $\mathrm{SL}(2,\mathbb{C})$ invariant vacuum of the closed string, it is precisely the boundary state (4.19) $|\mathrm{vac}\rangle = |\alpha\rangle$ describing the D9 brane.

Using this simple formal equivalence between constraints for superstrings on D-branes and Dirac-Kähler formalism on loop space a couple of further formal identifications are maybe interesting:

For the open string NS sector the last relation of (4.19) changes the sign

$$\left(\psi_r^\mu + i\bar{\psi}_{-r}^\mu\right)\left|\alpha'\right\rangle = 0\,, \qquad \forall\,r,\mu \;\text{ NS sector} \tag{4.27}$$

and now implies that the vacuum is, from the loop space perspective, the formal *volume form* instead of of the constant 0-form, i.e. that form annihilated by all differential form multiplication operators:

$$\partial_{(\mu,\sigma)} \left|\alpha'\right\rangle = 0 = \mathcal{E}^{\dagger(\mu,\sigma)} \left|\alpha'\right\rangle \qquad \forall\, \mu, \sigma\,. \tag{4.28}$$

In finite dimensional flat manifolds of course both are related simply by *Hodge duality*:

$$\left|\alpha'\right\rangle = \star \left|\alpha\right\rangle\,. \tag{4.29}$$

4.2.1 Loop space formalism

In the following it is demonstrated how the notation concerning differential geometry on loop space used in [14] translates to that used in [28, 29]. CIT: Schreiber:2004 CIT: Hofmann:2002,GetzlerJonesP

Consider a differential $p+1$ form ω on target space. It lifts to a $p+1$-form Ω on loop space given by

$$\Omega := \frac{1}{(p+1)!}\int\limits_{S^1} \omega_{\mu_1\cdots\mu_{p+1}}(X)\,\mathcal{E}^{\dagger\mu_1}\cdots\mathcal{E}^{\dagger\mu_{p+1}}\,. \tag{4.30}$$

Let $\hat{K} = X'^{(\mu,\sigma)}\mathcal{E}_{(\mu,\sigma)}$ be the operator of interior multiplication with the reparameterization Killing vector K on loop space. The above $p+1$-form is sent to a p-form $\oint(\omega)$ on loop space by contracting with this Killing vector:

$$\begin{aligned}\oint(\omega) &:= \left[\hat{K},\Omega\right]\\ &= \frac{1}{p!}\int\limits_{S^1} d\sigma\; \omega_{\mu_1\cdots\mu_{p+1}}\mathcal{E}^{\dagger\mu_1}\cdots\mathcal{E}^{\dagger\mu_p}X'^{\mu_{p+1}}\,.\end{aligned} \tag{4.31}$$

The anticommutator of the loop space exterior derivative $\mathbf{d}$ with $\hat{K}$ is just the reparameterization Killing Lie derivative

$$\left\{\mathbf{d},\hat{K}\right\} = \mathcal{L}_K \tag{4.32}$$

which commutes with 0-modes of fields of definite reparameterization weight, i.e.

$$[\mathcal{L},\Omega] = 0\,. \tag{4.33}$$

It follows that

$$\left[\mathbf{d},\left[\hat{K},\Omega\right]\right] = [\mathcal{L},\Omega] - \left[\hat{K},[\mathbf{d},\Omega]\right] \tag{4.34}$$ LAB: an equation

which implies that

$$\left[\mathbf{d},\oint(\omega)\right] = \oint(-d\omega)\,. \tag{4.35}$$

For the evaluation of generalized Wilson lines along the loops one needs to consider path ordered integrals. We deviate slightly from the definition in [28] and consider the CIT: Hofmann:2002 cyclically closed path ordererd integral, as it appears in the evaluation of Wilson loops (in particular making contact with equation (2.17) of [30]): CIT: Schreiber:2004b

$$\oint(\omega_1,\cdots,\omega_n) := \prod_i \frac{1}{(p_i+1)!} \int\limits_{\substack{0\leq\sigma^1\leq 2\pi \\ \sigma^{i-1}\leq\sigma^i\leq\sigma^1+2\pi}} d^n\sigma\ \left[\hat K,\omega_1\right](\sigma_1)\cdots\left[\hat K,\omega_n\right](\sigma_n)\ . \quad (4.36)$$

LAB: cyclic loop space int

For the special case that all the ω_n have the same degree, supercommutating $\mathbf{d}$ with these objects yields equation (5) of [28]: CIT: Hofmann:2002

$$\begin{aligned}
&\left[\mathbf{d}, \oint(\omega_1,\cdots,\omega_n)\right] \\
&= \sum_k (-1)^{\left.1+\sum_{i<k}p_i\right)} \oint(\omega_1,\cdots,d\omega_k,\cdots,\omega_n) \\
&\quad + \sum_k (-1)^{\left.\sum_{i<k}p_i\right)} \int\limits_{\substack{0\leq\sigma^1\leq 2\pi \\ \sigma^{i-1}\leq\sigma^i\leq\sigma^1+2\pi}} d^n\sigma\ \left[\hat K,\omega_1\right](\sigma_1)\cdots(\omega_k)'\cdots\left[\hat K,\omega_n\right](\sigma_n) \\
&= \sum_k (-1)^{\left.1+\sum_{i<k}p_i\right)} \oint(\omega_1,\cdots,d\omega_k,\cdots,\omega_n) \\
&\quad + \sum_k (-1)^{\left.1+\sum_{i<k}p_i\right)} \\
&\quad \int\limits_{\substack{0\leq\sigma^1\leq 2\pi \\ \sigma^{i-1}\leq\sigma^i\leq\sigma^1+2\pi}} d^n\sigma\ \left[\hat K,\omega_1\right](\sigma_1)\cdots\left(\left[\hat K,\omega_{k-1}\right]\omega_k - (-1)^{p_{k-1}}\omega_{k-1}\left[\hat K,\omega_k\right]\right)(\sigma_k)\cdots\left[\hat K,\omega_n\right](\sigma_{n-1}) \\
&= \sum_k (-1)^{\left.1+\sum_{i<k}p_i\right)} \left(\oint(\omega_1,\cdots,d\omega_k,\cdots,\omega_n) + \oint(\omega_1,\cdots,\omega_{k-1}\wedge\omega_k,\cdots,\omega_n)\right)\ . \quad (4.37)
\end{aligned}$$

LAB: cyclic loop

For the open string this should be generalized as follows:
Modify definition (4.36) is modified to

$$\oint(\omega_1,\cdots,\omega_n) := \prod_i \frac{1}{(p_i+1)!} \int\limits_{0\leq\sigma^1\leq\cdots\leq\sigma^n\leq\pi} d^n\sigma\ \left[\hat K,\omega_1\right](\sigma_1)\cdots\left[\hat K,\omega_n\right](\sigma_n)\ . \quad (4.38)$$

LAB: cyclic loop space int

Then (4.37) becomes

$$\begin{aligned}
&\left[\mathbf{d}, \oint(\omega_1,\cdots,\omega_n)\right] \\
&= \sum_k (-1)^{\left.1+\sum_{i<k}p_i\right)} \left(\oint(\omega_1,\cdots,d\omega_k,\cdots,\omega_n) + \oint(\omega_1,\cdots,\omega_{k-1}\wedge\omega_k,\cdots,\omega_n)\right) \\
&\quad -\omega_1(0)\oint(\omega_2,\cdots,\omega_n) + (-1)^{\left.\sum_{i<n}p_i\right)}\oint(\omega_1,\cdots,\omega_{n-1})\omega_n(\pi)\ .
\end{aligned} \quad (4.39)$$

This is proposition 1.6 in [29]. CIT: GetzlerJonesPetrack:1991

Consider cases where the boundary terms in the last line vanish. In that case one can collect terms in the path ordered exponential

$$\begin{aligned} U(a,b) &:= \mathbf{P}\exp\left(\int_a^b A\right) \\ U^{-1}(a,b) = U(b,a) &:= \mathbf{P}_{\rm inv}\exp\left(-\int_a^b A\right) \end{aligned} \tag{4.40}$$

over the 1-form A:

$$\begin{aligned} \left[\mathbf{d}, \mathbf{P}\exp\left(\int\limits_0^\pi A\right)\right] &= \left[\mathbf{d}, \sum_{n=0}^\infty \oint \underbrace{(A,\cdots,A)}_{n \text{ times}}\right] \\ &= -\sum_{n=0}^\infty \sum_k \oint (A,\cdots,A,\mathbf{d}_A A, A, \cdots A)_{n \text{ occurences of } A,\ \mathbf{d}_A A \text{ at } k} \\ &= -\int\limits_0^\pi d\sigma\, \mathbf{P}\exp\left(\int\limits_0^\sigma A\right)\left[\hat K, \mathbf{d}_A A\right](\sigma)\, \mathbf{P}\exp\left(\int\limits_\sigma^\pi A\right)\,. \end{aligned} \tag{4.41}$$

This implies that

$$U(\pi,0)\circ \mathbf{d}_K \circ U(0,\pi) = \mathbf{d}_K - \int\limits_0^\pi d\sigma\, U(\pi,\sigma)\left[\hat K, \mathbf{d}_A A\right](\sigma)\, U(\sigma,\pi)\,. \tag{4.42}$$

LAB: lextd deformed by W

4.2.2 On BSCFT deformations in loop space

LAB: On classical BSCFT deformations in loop space

In [14] the deformation technique reviewed in §4.1 (p.22) was studied in functional loop space formalism and, as a first approximation, without taking normal ordering effects into account. It was shown that this way some information about superstrings in non-trivial background can be obtained, but of course some quantum effects, like the equations of motion of the background fields, require to properly take normal ordering into account. CIT: Schreiber:2004

In this section we remark on some aspects of the application of the formalism used in [14] to open strings. CIT: Schreiber:2004

In the case of closed strings it was shown that all massless NS backgrounds can be obtained by using deformation operators A (4.8) of the form $A = \exp\left(\oint w(\sigma)\right)$, where $w(\sigma)$ is of unit spatial reparameterization weight.

For the open string it is possible to replace the full closed contour integral $\oint$ by one over the bounded open string $\oint \to \int\limits_0^\pi$ if the field $w(\sigma)$ *vanishes at the string's endpoints*:

$$w(\pi) \stackrel{!}{=} 0 \stackrel{!}{=} w(0)\,. \tag{4.43}$$

LAB: boundary condition

With this condition $\mathbf{W}$ is still reparameterization invariant:

$$\left[\mathcal{L}_\xi, \int\limits_0^\pi d\sigma\; w(\sigma)\right] = \int\limits_0^\pi d\sigma\; \left(\xi(\sigma)\, w(\sigma)\right)'$$

$$
\begin{aligned}
&= \xi(\sigma)\, w(\sigma)|_{\sigma=0}^{\sigma=\pi} \\
&\overset{(4.43)}{=} 0
\end{aligned} \tag{4.44}
$$

as required by (4.7).

For example consider the open string on the space-filling D-brane with Neumann boundary condition in all directions, so that

$$
\begin{aligned}
X'^{\mu}(\pi) = 0 = X'^{\mu}(0) \qquad \forall\, \mu \\
\mathcal{E}^{\dagger\mu}(0) = 0 = \mathcal{E}^{\dagger\mu}(\pi)
\end{aligned} \tag{4.45}
$$

LAB: Neumann condition

(this is just the T-dual of (4.26), describing the transverse propagation of an open string instead of the equivalent longitudinal propagation of a closed string) and consider a shift of the background gauge field A_μ, which, following equation (3.48) of [14] is induced by

CIT: Schreiber:2004

$$
\mathbf{W} = \int_0^\pi d\sigma\, \left(iA_\mu(X(\sigma))\, X'^{\mu}(\sigma) + \frac{1}{2T}(dA)_{\mu\nu}(X(\sigma))\, \mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu} \right) . \tag{4.46}
$$

The deformed loop space exterior derivative then reads

$$
\begin{aligned}
& e^{-\mathbf{W}} \mathbf{d}_K(\sigma)\, e^{\mathbf{W}} \\
&= \mathbf{d}_K(\sigma) + i \int_0^\pi d\sigma' \, \left(\mathcal{E}^{\dagger\mu}(\sigma) \left((\partial_\mu A_\nu)(\sigma')\, X'^{\nu}(\sigma')\, \delta(\sigma - \sigma') + A_\mu(\sigma')\, \delta'(\sigma' - \sigma) \right) - (dA)_{\mu\nu}(\sigma')\, \mathcal{E}^{\dagger\mu} X'^{\nu} \right) \\
&= \mathbf{d}_K(\sigma) + i\mathcal{E}^{\dagger\mu}(\sigma)\, A_\mu(\sigma)\, (\delta(\sigma - \pi) - \delta(\sigma)) \, .
\end{aligned} \tag{4.47}
$$

The bulk terms mutually cancel and leave only the usual coupling of the endpoints of the string to the gauge field.

(*cf.* [25, 26, 27]

CIT: MaedaNakatsuOonishi:2004,Hashimoto:2000

There is a natural generalization to non-Abelian gauge groups. Let

$$
\exp(\mathbf{W}) = \mathbf{P} \exp\left(\int_0^\pi \left(iA_\mu X'^{\mu} + \frac{1}{2}(\frac{1}{T} d_A A + B)_{\mu\nu} \mathcal{E}^{\dagger\mu}\mathcal{E}^{\dagger\nu} \right) \right) . \tag{4.48}
$$

This is manifestly still reparameterization invariant, for the same reasons as before.

When restricting attention to the terms of rank one (which contain the connection part) one finds, using (4.42)

$$
\begin{aligned}
\exp(-\mathbf{W}) \circ \mathbf{d}_K \circ \exp(\mathbf{W}) = {} & \mathbf{d}_K \\
& + i\mathcal{E}^{\dagger\mu}(\sigma)\, U(\pi,\sigma)\, A_\mu(\sigma)\, U(\sigma,\pi)\, (\delta(\sigma - \pi) - \delta(\sigma)) \\
& + iT \int_0^\pi d\sigma\; U(\pi,\sigma) \left[\hat{K}, B \right](\sigma)\, U(\sigma,\pi) \\
& + \mathcal{O}(\mathcal{E}^2)
\end{aligned} \tag{4.49}
$$

LAB: non-abelian B-field

The second line is the A-field connection on the Chan-Paton factors. The third line then apparently is the non-Abelian B-field connection. It has a nice heuristic interpretation:

For every point σ in the bulk the Chan-Paton factor is first parallel transported from the end of the string to σ using the connection A, there is is multiplied with the non-Abelian B-field at that point and then again parallel transported back to the end of the string by A.

By comparison with equation (3.42) in [14] one readily sees that the consistency condition (3.24)[14] is satisfied. The only new thing to check is that the holonomy terms $U(\cdot,\cdot)$ don't produce problematic terms. Using the abbreviation $R = i\mathcal{E}^{\dagger\mu}A_\mu(\delta_\pi - \delta_0) + iT\left[\hat{K}, B\right]$ and concentrating on modex $\xi_1(\sigma)$ and $\xi_2(\sigma)$ which vanish at $\sigma \in \{0,\pi\}$ we have CIT: Schreiber:2004 CIT: Schreiber:2004

$$\begin{aligned}
\left[\mathbf{d}^{(A,B)}_{K,\xi_1}, \mathbf{d}^{\dagger(A,B)}_{K,\xi_2}\right] &= \int d\sigma\; \xi_1\xi_2(\cdots) \\
&\quad + \left[\mathbf{d}^{(A,B)}_{K,\xi_1}, \int U(\pi,\sigma)\xi_2 R^\dagger(\sigma)\, U(\sigma,\pi)\right] + \left[\mathbf{d}^{\dagger(A,B)}_{K,\xi_1}, \int U(\pi,\sigma)\xi_1 R(\sigma)\, U(\sigma,\pi)\right] \\
&= \int d\sigma\; \xi_1\xi_2(\cdots) \\
&\quad + \int\!\!\int_{\sigma'<\sigma} d\sigma\; d\sigma'\; U\big(\pi,\sigma'\big)\,\xi_1 R\big(\sigma'\big)\,U\big(\sigma',\sigma\big)\,\xi_2 R^\dagger(\sigma)\,U(\sigma,\pi) \\
&\quad - \int\!\!\int_{\sigma'>\sigma} d\sigma\; d\sigma'\; U(\pi,\sigma)\,\xi_2 R^\dagger(\sigma)\,U\big(\sigma,\sigma'\big)\,\xi_1 R\big(\sigma'\big)\,U\big(\sigma',\pi\big) \\
&\quad + \int\!\!\int_{\sigma'>\sigma} d\sigma\; d\sigma'\; U(\pi,\sigma)\,\xi_2 R^\dagger(\sigma)\,U\big(\sigma,\sigma'\big)\,\xi_1 R\big(\sigma'\big)\,U\big(\sigma',\pi\big) \\
&\quad - \int\!\!\int_{\sigma'<\sigma} d\sigma\; d\sigma'\; U\big(\pi,\sigma'\big)\,\xi_1 R\big(\sigma'\big)\,U\big(\sigma',\sigma\big)\,\xi_2 R^\dagger(\sigma)\,U(\sigma,\pi) \\
&= \int d\sigma\; \xi_1\xi_2(\cdots)\,.
\end{aligned} \tag{4.50}$$

LAB: consistency

This means that the deformation (4.49) is indeed a consistent superconformal deformation (At the classical, level, as usual. Consistency at the quantum level will lead to the equations of motion for the A and B field as demonstrated in (*cf.* [26])) and hence describes the propagation of open superstrings on a stack of D9 branes with a gauge field A and a Kalb-Ramond-field B turned on, both taking values in some $N \times N$ matrix algebra. CIT: Hashimoto:2000

This also demonstrates that the gauge covariant loop space exterior derivative (4.49) is, at least for our purposes, preferable over that proposed in [28], which is missing the right $U(\sigma,\pi)$ factor in the third line of (4.49), which is precisely the term needed in order for the problematic terms in (4.50) to vanish. CIT: Hofmann:2002

Further literature: Non-Abelian 2-forms have been discussed for instance in [31] in the context of field theory and in [32] the context of p-gerbes, in [28] in loop space formalism. CIT: Lahiri:2001 CIT: Zunger:2000 CIT: Hofmann:2002

4.2.3 Background equations of motion

The above objects on loop space won't be well defined in general due to divergences. These should vanish when the background fields satisfy their equations of motion. For the case of the ablelian A-field this has been checked in [26]. Using the above formalism we can easily derive the consistency condition in the nonabelian case if we restrict attention to *constant* gauge connections. This greatly simplifies the calculation, since the only worldsheet field whose divergences are to be calculated is the X' coming from the $A_\mu X'^\mu$. CIT: Hashimoto:2000

A glance at (4.49) shows that divergences in this expression to first order comes from $X'X'$ contractions associated with $[A_\mu, [A^\mu, A_\nu]]$ and $[A_\mu, B^\mu{}_\nu]$.

4.2.4 Boundary DDF/Pohlmeyer invariants

The Pohlmeyer invariant [30] $Z_+(A)$ is the Wilson line of a the *constant* gauge fields A over the closed string with respect to $\mathcal{P}_+^\mu = \frac{1}{\sqrt{2T}}(P + iTX')^\mu$: CIT: Schreiber:2004b

$$Z_+(A) := \mathrm{Tr}\,\mathbf{P}\exp\left(\oint d\sigma\; A_\mu \mathcal{P}_+^\mu(\sigma)\right) . \tag{4.51}$$

LAB: Pohlmeyer invariant

It is well known [25, 26, 27] that the similar Wilson line $U(A)$ with respect to X' CIT: MaedaNakatsuOonishi:2004,Hashimoto:2000

$$U(A) := \mathrm{Tr}\,\mathbf{P}\exp\left(\oint d\sigma\; A_\mu X'^\mu(\sigma)\right) \tag{4.52}$$

is the operator which sends the boundary state $|\alpha\rangle$ of a bare D-brane to the state

$$|\alpha(A)\rangle := U\,|\alpha\rangle \tag{4.53}$$

describing the same D-brane but with the gauge field A turned on. (Here everything applies to the tangential directions of the D-brane.)

But the bare boundary state $|\alpha\rangle$ has the defining property

$$\mathcal{P}_+^\mu(\sigma)\,|\alpha\rangle = \sqrt{\frac{T}{2}} X'^\mu(\sigma)\,|\alpha\rangle\;, \tag{4.54}$$

which suggests that applied to the boundary state the action of the Pohlmeyer invariant is actually the same as that of the ordinary Wilson loop $U(A)$:

$$Z_+\left(\sqrt{2/T}A\right)|\alpha\rangle = U(A)\,|\alpha\rangle\;. \tag{4.55}$$

LAB: Pohlmeyer/Wilson b

The interesting thing about this equivalence is that the left hand side is in a sense better behaved than the right hand side because $Z_+(A)$ commutes with *all* Virasoro constraints (or all super-Virasoro constraints when appropriately generalized as below), while $U(A)$ only commutes with the (super) reparameterization constraint with respect to the spatial worldsheet coordinate σ. This means in particular that the BRST invariance of the left hand side is manifest and in particular manifest for arbitrary BRST invariant states $|\alpha\rangle$. This is helpful when in addition to the gauge field further background fields are to be turned on.

We now show that (4.55) is indeed true whenever one of the two sides is well defined (which, according to [26, 27], should be the case iff A satisfies its background equations of motion). CIT: Hashimoto:2000,Hashimo

Actually, there is a simple argument which demonstrates the truth of (4.55) in one line. This consists in observing that the ordering of the operators $\mathcal{P}^\mu$ in (4.51) is arbitrary[5]:

When commuting any two operators $\mathcal{P}$ in the Pohlmeyer invarint we produce a δ' distribution and due to cyclic invariance we may always identify its arguments with σ^1 and σ^2 in formula (2.17) of [30]. Integration over these two variables then shows that the result vanishes. CIT: Schreiber:2004b

The problem with this simple argument is that it is not easy to see if it can be spoiled by quantum anomalies due to the infinite sums that are involved (when translating to a Fourier representation).

For this reason, and because it yields a better understanding of some interesting related phenomena, in the following a more detailed analysis of (4.55) is given. The key idea is to generalize the technique used in [30] and express the Wilson loop $U(A)$ in terms of reparameterization invariant quasi-local substitutes of X': CIT: Schreiber:2004b

For technical reasons we will need to assume that the brane is wrapped along a lightlike direction. For this reason briefly recall the notation of [30] generalized so as to incorporate non-trivial winding modes. CIT: Schreiber:2004b

With $X^\mu(\sigma)$ and $P_\mu(\sigma)$ the canonical coordinates and momenta of the string we define

$$\mathcal{P}^\mu_\pm(\sigma) = \frac{1}{\sqrt{2T}}\left(P^\mu(\sigma) \pm T X'^\mu(\sigma)\right) \tag{4.56}$$

with mode expansion

$$\begin{aligned}
\mathcal{P}^\mu_+(\sigma) &:= \frac{1}{\sqrt{2\pi}}\sum_m \tilde{\alpha}^\mu_m e^{-im\sigma} \\
\mathcal{P}^\mu_-(\sigma) &:= \frac{1}{\sqrt{2\pi}}\sum_m \alpha^\mu_m e^{+im\sigma}
\end{aligned} \tag{4.57}$$

which can be used to define the non-local fields

$$\begin{aligned}
X^\mu_-(\sigma) &:= x^\mu_- - \frac{\sigma}{4\pi T}p^\mu_- + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\alpha^\mu_m e^{+im\sigma} \\
X^\mu_+(\sigma) &:= x^\mu_+ + \frac{\sigma}{4\pi T}p^\mu_+ + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\tilde{\alpha}^\mu_m e^{-im\sigma}
\end{aligned} \tag{4.58}$$

that satisfy

$$\begin{aligned}
\left(X^\mu_-\right)'(\sigma) &= -\frac{1}{\sqrt{2T}}\mathcal{P}^\mu_-(\sigma) \\
\left(X^\mu_+\right)'(\sigma) &= \frac{1}{\sqrt{2T}}\mathcal{P}^\mu_+(\sigma)
\end{aligned} \tag{4.59}$$

LAB: Xpm prime

[5] This does *not* mean that the path ordering is not essential, just that the classical path ordered expression has no operator ordering ambiguity when quantized. Path ordering here determines the pairing of the spacetime indices carried by $\mathcal{P}^\mu(\sigma)$ with the corresponding worldsheet coordinate σ. This is not related to the ordering of the $\mathcal{P}^\mu(\sigma)$ among themselves.

and sum to X:

$$\begin{aligned} X^\mu(\sigma) &= X_+^\mu(\sigma) + X_-^\mu(\sigma) \\ &= x_+^\mu + x_-^\mu + \frac{\sigma}{\sqrt{4\pi T}}\left(p_+^\mu - p_-^\mu\right) + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\frac{1}{m}\left(\alpha_m^\mu - \tilde{\alpha}_{-m}^\mu\right)e^{+im\sigma} . \end{aligned} \tag{4.60}$$

The spatial derivative of X has the expansion

$$X'^\mu(\sigma) = \frac{1}{\sqrt{4\pi T}}\left(p_+^\mu - p_-^\mu\right) + \frac{i}{\sqrt{4\pi T}}\sum_{m\neq 0}\left(\alpha_m^\mu - \tilde{\alpha}_{-m}^\mu\right)e^{+im\sigma} . \tag{4.61}$$

Our boundary state $|\alpha\rangle$ encodes Neumann boundary conditions

$$\left(\alpha_n^\mu + \bar{\alpha}_{-n}^\mu\right)|\alpha\rangle = 0 \tag{4.62}$$

and the above shows that for the left/right center-of-mass coordinates we should set

$$\left(x_+ - x_-\right)|\alpha\rangle \stackrel{!}{=} 0 \tag{4.63}$$

so that the total condition on the boundary state can be summarized as

$$\left(X_+^\mu(\sigma) - X_-^\mu(\sigma)\right)|\alpha\rangle = 0 , \tag{4.64}$$

which implies

$$X^\mu(\sigma)\,|\alpha\rangle = 2X_+^\mu(\sigma)\,|\alpha\rangle = 2X_-^\mu(\sigma)\,|\alpha\rangle . \tag{4.65}$$

The canonical momentum has the expansion

$$\begin{aligned} P^\mu(\sigma) &= \sqrt{\frac{T}{4\pi}}\sum_m\left(\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\right)e^{-im\sigma} \\ &= \frac{1}{4\pi}\left(p_+^\mu + p_-^\mu\right) + \sqrt{\frac{T}{4\pi}}\sum_{m\neq 0}\left(\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\right)e^{+im\sigma} \end{aligned} \tag{4.66}$$

and the canonical commutation relations imply

$$\begin{aligned} \left[x_\pm^\mu, p_\pm^\mu\right] &= i \\ \left[x_\pm^\mu, p_\mp^\mu\right] &= 0 . \end{aligned} \tag{4.67}$$

We write the worldsheet fermions as Clifford generators $\Gamma_\pm = \mathcal{E}^\dagger \pm \mathcal{E}$ (4.22) on loop space:

$$\begin{aligned} \Gamma_-^\mu(\sigma) &= -\frac{i}{\sqrt{\pi}}\sum_r b_r^\mu e^{ir\sigma} \\ \Gamma_+^\mu(\sigma) &= +\frac{1}{\sqrt{\pi}}\sum_r \tilde{b}_r^\mu e^{-ir\sigma} , \end{aligned} \tag{4.68}$$

and their boundary condition is (4.26)

$$\left(\Gamma^\mu_+(\sigma) - \Gamma^\mu_-(\sigma)\right)|\alpha\rangle = 0\,. \tag{4.69}$$

Now recall that the DDF invariants are obatined by supercommuting the supercharge with the integral over a weight 1/2 field:

$$\begin{aligned} A^\mu_m &:= \left\{G_0, \frac{i}{\sqrt{4\pi}}\oint d\sigma\; \Gamma^\mu_-(\sigma)\, e^{-imR_-(\sigma)}\right\} \\ &= \frac{1}{\sqrt{2\pi}}\oint d\sigma\; \left(\mathcal{P}^\mu(\sigma) - m\frac{2\pi\sqrt{2T}}{k\cdot p_-}k\cdot\Gamma_-(\sigma)\,\Gamma^\mu_-(\sigma)\right) e^{-imR_-(\sigma)}\,, \end{aligned} \tag{4.70}$$

LAB: a DDF invariant

where k is a fixed but arbitrary lightlike vector in target space.

This is straightforwardly generalized to the σ-reparameterization super subalgebra of the total constraint algebra. Assume nontrivial winding in the k-direction so that $k\cdot(p_+ - p_-) = -2k\cdot p_-$ is non-vanishing at the boundary state.

Recall that the deformed loop space exterior derivative (4.24) is defined as

$$\mathbf{d}_K := \sqrt{T}\left(G_\nu + i\tilde{G}_\nu\right) \tag{4.71}$$

and define in generalization of equation (2.42) of [30] the reparameterization field

CIT: Schreiber:2004b

$$R(\sigma) = -\frac{4\pi T}{k\cdot(p_+ - p_-)}k\cdot X(\sigma)\;. \tag{4.72}$$

Using all this the analogue of (4.70) is

$$\begin{aligned} \mathcal{X}'^\mu_n &:= \left\{\mathbf{d}_K, \frac{-i}{\sqrt{4\pi T}}\left(\oint d\sigma\; e^{in\nu}\mathcal{E}^{\dagger\mu}(\sigma)\exp(irR(\sigma))\right)\right\} \\ &= \oint d\sigma\; \left(\sqrt{T/4\pi}X'^\mu(\sigma) - r\frac{\sqrt{4\pi T}}{k\cdot(p_+ - p_-)}k\cdot\mathcal{E}^\dagger(\sigma)\,\mathcal{E}^{\dagger\mu}(\sigma)\right)\exp(irR(\sigma)) \\ &= \oint d\sigma\; \left(\sqrt{T/4\pi}X'^\mu(\sigma) - r\frac{\sqrt{4\pi T}}{k\cdot(p_+ - p_-)}k\cdot\mathcal{E}^\dagger(\sigma)\,\mathcal{E}^{\dagger\mu}(\sigma)\right)\exp(inR(\sigma)) \\ &\overset{(4.59)}{=} \frac{1}{2}\frac{1}{\sqrt{2\pi}}\oint d\sigma\; \left(\mathcal{P}^\mu_+(\sigma) - \mathcal{P}^\mu_-(\sigma) - r\frac{2\pi\sqrt{2T}}{k\cdot(p_+ - p_-)}k\cdot\mathcal{E}^\dagger(\sigma)\left(\Gamma^\mu_+ + \Gamma^\mu_-\right)(\sigma)\right)\exp(inR(\sigma))\;. \end{aligned} \tag{4.73}$$

(This requires that we are in the Ramond sector, because otherwise phase offsets $e^{\pm i\nu}$ would be required for the left/right moving sector, which is not compatible.)

By construction, following the same ideas as for the ordinary DDF invariants, these objects are super σ-reparameterization invariant:

$$\begin{aligned} \left[\mathbf{d}_{K,n}, \mathcal{X}'^\mu_m\right] &= 0 \\ \left[\mathcal{L}_n, \mathcal{X}'^\mu_m\right] &= 0\,. \end{aligned} \tag{4.74}$$

For our purposes another crucial property is

$$\begin{aligned} \mathcal{X}'^{\mu}_n \left|\alpha\right\rangle &= -\frac{1}{\sqrt{2\pi}} \oint d\sigma \; \left(\mathcal{P}^{\mu}_{-}(\sigma) + r\frac{2\pi\sqrt{2T}}{k\cdot p_-} k\cdot\Gamma_-(\sigma)\,\Gamma^{\mu}_-(\sigma)\right) \exp(inR_-(\sigma))\left|\alpha\right\rangle \\ &= -A^{\mu}_{-n}\left|\alpha\right\rangle \\ &= \tilde{A}^{\mu}_n \left|\alpha\right\rangle \end{aligned} \tag{4.75}$$

as well as

$$[\mathcal{X}^{\mu}_n, A^{\nu}_m]\cdots\left|\alpha\right\rangle = \left[-A^{\mu}_{-n}, A^{\nu}_m\right]\cdots\left|\alpha\right\rangle \,, \tag{4.76}$$

where the ellipsis indicates further arbitrary terms that have non-vanishing commutators with oscillators in the k direction.

These relations imply that arbitrary products of $\mathcal{X}'_n$ on the boundary state $\left|\alpha\right\rangle$ can be replaced by their A_n cousins:

$$\mathcal{X}^{\mu_1}_{n_1}\mathcal{X}^{\mu_2}_{n_2}\cdots\mathcal{X}^{\mu_N}_{n_N}\left|\alpha\right\rangle = A^{\mu_1}_{n_1}A^{\mu_2}_{n_2}\cdots A^{\mu_3}_{n_3}\,. \tag{4.77}$$

LAB: replacing cal X by A

(All this holds true at the quantum level without problems at least for transversal operators. Longitudinal excitation require further examination.)

It remains to express the Wilson line and the Pohlmeyer invariant in terms of the DDF-like operators. This follows directly the construction in [30]. The quasi-local version of the $\mathcal{X}'_n$ is

CIT: Schreiber:2004b

$$\mathcal{X}'^{\mu}(\sigma) := \frac{1}{\sqrt{2\pi}}\sum_n \mathcal{X}'^{\mu}_n e^{-in\sigma} \tag{4.78}$$

and by the same argument as for the Pohlmeyer invariants the Wilson line is unaffected when all $X'(\sigma)$ are replaces by $\sqrt{2/T}\mathcal{X}'(\sigma)$. This reduces the entire Wilson line to a linear combination of terms of the form (4.77), which finally implies (4.55).

A. Review of BRST methods

LAB: Review of BRST methods

A.1 The exterior derivative on the gauge group

The exterior derivative $\mathbf{d}$ on any manifold $(\mathcal{M}, g)$ is

$$\mathbf{d} = \hat{c}^{\dagger\mu} \partial_\mu \,, \tag{A.1}$$

where $\hat{c}^\dagger$ is the operator of exterior multiplication with dx^μ and ∂_μ is the partial derivative such that $\left[\hat{c}^{\dagger\mu}, \partial_\nu\right] = 0$. In terms of vielbein field $e_a = e_a{}^\mu$ this can be rewritten as (see appendix A.2 of [33])

CIT: Schreiber:2003a

$$\mathbf{d} = \hat{e}^{\dagger a} t_a - \omega_a{}^b{}_c \hat{e}^{\dagger a} \hat{e}^{\dagger c} \hat{e}_b \,, \tag{A.2}$$

where $\hat{e}^{\dagger a} = e^a{}_\mu \hat{c}^{\dagger\mu}$, $\hat{e}_a = e_a{}^\mu \hat{c}^\mu$ are the orthonormal form creators/annihilators, t_a is the differential operator satisfying $[t_a, f] = e_a{}^\mu \partial_\mu f$ for 0-forms f and $\left[t_a, \hat{e}^{\dagger b}\right] = 0 = [t_a, \hat{e}_b]$, and ω is the Levi-Civita connection in the ONB frame.

For $(\mathcal{M}, g)$ a Lie group, t_a the left/right invariant vector fields and $f_a{}^c{}_b$ the structure constants such that

$$[t_a, t_b] = f_a{}^c{}_b \, t_c \tag{A.3}$$

we have

$$\omega_{a}{}^{b}_{c} = \frac{1}{2} f_a{}^b{}_c \tag{A.4}$$

(e.g. appendix D of [33]) and hence

CIT: Schreiber:2003a

$$\mathbf{d} = \hat{e}^{\dagger a} t_a - \frac{1}{2} f_a{}^b{}_c \hat{e}^{\dagger a} \hat{e}^{\dagger c} \hat{e}_b \,. \tag{A.5}$$

LAB: extd on Lie group

Anticommuting $\mathbf{d}$ with $\hat{c}_a$ yields the Lie derivative $\mathcal{L}_{e_a}$ along the vector field e_a on differential forms:

$$\begin{aligned} \mathcal{L}_{e_a} &= \{\mathbf{d}, \hat{c}_a\} \\ &= t_a - f_a{}^b{}_c \hat{e}^{\dagger c} \hat{e}_b \,. \end{aligned} \tag{A.6}$$

LAB: Lie derivative

The second term constitutes another representation of the Lie algebra, so that

$$[\mathcal{L}_a, \mathcal{L}_b] = f_a{}^c{}_b \, \mathcal{L}_c \,. \tag{A.7}$$

Now suppose that the Lie group in question is the gauge group of some physical system, t_a are the generators of gauge transformations and physical states $|\psi\rangle$ are functions on $(\mathcal{M}, g)$, taking values in some appropriate space. Then the conditions that these states be gauge invariant is that they are annihilated by the generators t_a:

$$t_a \, |\psi\rangle = 0 \,, \qquad \forall \, a \,. \tag{A.8}$$

For $|\psi\rangle$ a 0-form this set of equations is concisely summarised by the single equation

$$\mathbf{d} \, |\psi\rangle = 0 \,. \tag{A.9}$$

LAB: gauge invariance as

It is natural to also consider $(p > 0)$-forms on $(\mathcal{M}, g)$ and to apply the notion (A.9) of gauge invariance to all of these. In this case the operator $i\mathbf{d}$ is known as the *BRST operator*.

Following the presentation in section 4.2 of [34] the same can be understood in terms of the path integral as follows:

CIT: Polchinski:1998

A.2 The Fadeev-Popov method

An action $S = S(\phi)$ with gauge invariances δ_a with $[\delta_a, \delta_b] = f_a{}^b{}_c\,\delta_c$ has the (Euclidean) path integral (partition sum)

$$Z := \int \frac{D\phi}{V_{\text{gauge}}} e^{-S(\phi)} \tag{A.10}$$

where we integrate over all field configurations ϕ and divide out by the volume V_{gauge} of the gauge group to get rid of the overcounting due to gauge equivalent field configurations.

Equivalently, this means that we should integrate over a 'gauge slice', i.e. pick a representative element from each gauge orbit of the gauge group and integrate over these.

So let

$$F^A(\phi) = 0 \tag{A.11}$$

be the gauge conditions (where the index A will in general take continuous values). Then the path integral can be written as

$$\begin{aligned} Z &= \int \frac{d\phi}{V_{\text{gauge}}} e^{-S(\phi)} \\ &= \int e^{-S(\phi)} d\phi|_{F^A(\phi)=0} \,. \end{aligned} \tag{A.12}$$

Now divide the integration over all fields ϕ into an integration over an arbitrary representative $\hat\phi$ from each gauge orbit and an integration over the elements $g \in G$ of the gauge group, so that $\phi = g(\hat\phi)$:

$$Z = \int e^{-S(g(\hat\phi))} (d\hat\phi)(dg)|_{F^A\left(g(\hat\phi)\right)=0} \,. \tag{A.13}$$

This way it is transparent that the restriction to the gauge orbit can be expressed by insertion of $\delta(F(\phi))$ together with a prefactor[6] $\det\left(\delta_\alpha F^A\right)$, the *Fadeev-Popov determinant*, which accounts for the fact that we are integrating over g instead of over F:

$$\begin{aligned} Z &= \int (d\hat\phi)(dg) \det\left(\delta_\alpha F^A\right) \delta(F(g(\hat\phi)))\, e^{-S(g(\hat\phi))} \\ &= \int d\phi\ \det\left(\delta_\alpha F^A\right) \delta(F(\phi))\, e^{-S(\phi)} \,. \end{aligned} \tag{A.16}$$

[6]In more detail, this prefactor can be understood as follows:
The Haar measure on the gauge group is

$$dg = \prod_\alpha \sigma^\alpha \,, \tag{A.14}$$

where $\sigma^\alpha = e^\alpha{}_\mu dx^\mu$ are the invariant 1-forms on the group. We can assume that the functions F^A are chosen in such a way that there is an open neighborhood of $F = 0$ where these provide a set of good coordinates. Then the measure that restricts the interation to the gauge slice is

$$\begin{aligned} \prod_A dF^A\ \delta\ \ F^A(g(\hat\phi)) &= \prod_\alpha \sigma^\alpha \det(\delta_\alpha \mathrm{F}^\mathrm{A})\ \delta\ \ F^A(g(\hat\phi)) \\ &= dg \det(\delta_\alpha \mathrm{F}^\mathrm{A})\ \delta\ \ F^A(g(\hat\phi)) \quad . \end{aligned} \tag{A.15}$$

This now is the path integral over the unrestricted set of fields ϕ but with a new weight factor, i.e. a new action functional.

In order to make this more manifest one now reexpresses the new factors $\det(\delta_\alpha F^A)$ and $\delta(F(\phi))$ in terms of integrals over exponentials themselves. For the delta distribution we can write

$$\delta(F(\phi)) \propto \int dB \, e^{-iB_A F^A(\phi)} . \tag{A.17}$$

For the determinant one can use a Berezin integral over the Grassmann *ghost* fields c^α and b_A:

$$\det(\delta_\alpha F^A) \propto \int dc \, db \, \exp\left(b_A c^\alpha \, \delta_\alpha F^A(\phi)\right) . \tag{A.18}$$

This way the partition function becomes

$$Z \propto \int d\phi \, dB \, db \, dc \, e^{-S - S_{\mathrm{gf}} - S_{\mathrm{FP}}} , \tag{A.19}$$

where

$$S_{\mathrm{gf}} := -iB_A F^A(\phi) \tag{A.20}$$

is the gauge fixing action and

$$S_{\mathrm{FP}} := b_A c^\alpha \, \delta_\alpha F^A(\phi) \tag{A.21}$$

the Fadeev-Popov action.

Even though one has just rewritten the original path integral, the new auxiliary action $S + S_{\mathrm{gf}} + S_{\mathrm{FP}}$ has a remarkable symmetry under the following, so-called BRST (Becchi-Rouet-Stora-Tyutin) transformation:

$$\begin{aligned} \delta_{\mathrm{B}}\phi &= -i\epsilon c^\alpha \delta_\alpha \phi \\ \delta_{\mathrm{B}} B_A &= 0 \\ \delta_{\mathrm{B}} b_A &= \epsilon B_A \\ \delta_{\mathrm{B}} c^\alpha &= \epsilon \frac{i}{2} f_\beta{}^\alpha{}_\gamma . \end{aligned} \tag{A.22}$$

In these equations the quantity B can usually be expressed in terms of the other fields by using the equations of motion of ϕ. One finds that the conserved charge Q which generates these transformations (the BRST charge) is noting but the exterior derivative (A.5):

$$Q = i\mathbf{d} . \tag{A.23}$$ LAB: BRST operator

A.3 BRST quantization of the string

In the case of the bosnonic string the gauge group is the conformal group generated by the Virasoro generators $\{L_m\}$. Their structure constants (classically) follow from

$$[L_m, L_n] = (m - n)L_{m+n} \tag{A.24}$$

as

$$f_m{}^q{}_n = \delta_{q,m+n}(m-n)\,. \tag{A.25}$$

With the notation

$$\begin{aligned} c_m &:= \hat{\mathrm{e}}^{\dagger L_{-m}} \\ b_m &:= \hat{\mathrm{e}}_{L_m} \end{aligned} \tag{A.26}$$

for the ghosts/differential forms we have

$$\begin{aligned} \{c_m, b_n\} &= \left\{\hat{\mathrm{e}}^{\dagger -m}, \hat{\mathrm{e}}_n\right\} \\ &= \delta_{n,-m} \end{aligned} \tag{A.27}$$

and the BRST operator (A.23) reads

$$Q = \sum_{n=-\infty}^{\infty} c_n L_{-n} + \sum_{m,n} \frac{(m-n)}{2} :c_m c_n b_{-m-n}: + c_0 a\,. \tag{A.28}$$

The real number a is the normal ordering constant of L_0.

The *total* Virasoro generators are the Lie derivatives (A.6) with respect to the Virasoro group:

$$\begin{aligned} L_m^{\mathrm{tot}} &:= \{Q, b_m\} \\ &= \underbrace{L_m + a\delta_{m,0}}_{:=L_m^{\mathrm{matter}}} + \underbrace{\sum_{n=-\infty}^{\infty} (m-n) :b_{m+n} c_{-n}:}_{:=L_m^{\mathrm{ghost}}}\,. \end{aligned} \tag{A.29}$$

The algebra of these operators has in general an anomaly

$$\left[L_m^{\mathrm{tot}}, L_n^{\mathrm{tot}}\right] = (m-n)L_{m+n}^{\mathrm{tot}} + \delta_{m+n,0}A(m) \tag{A.30}$$

with

$$A(m) = \underbrace{\frac{D}{12}(m^3-m) - 2ma}_{:=A^{\mathrm{matter}}(m)} + \underbrace{\frac{1}{6}(m-13m^3)}_{:=A^{\mathrm{ghost}}(m)} \tag{A.31}$$

with a contribution from the 'matter' and from the ghost sector. Due to this anomaly the BRST charge is not necessarily nilpotent as in the classical case. It can be checked that the BRST operator only squares to 0 if $A(m) = 0$ which is the case presicely for $D = 26$ and $a = -1$, in which case the theory is conformally invariant also at the quantum level.

A.4 Superstring BRST theory

A.4.1 Pictures

$$\begin{aligned} c &= e^{\sigma} \\ b &= e^{-\sigma} \end{aligned} \tag{A.32}$$

super-reparameterization ghosts:

$$\begin{aligned} \gamma &= e^{\phi}\eta \;=\; e^{\phi-\chi} \\ \beta &= e^{-\phi}\partial\xi \;=\; e^{-\phi}\rho \;=\; e^{-\phi+\chi}\partial\chi \end{aligned} \tag{A.33}$$

$$\begin{aligned} \mathcal{O}(z) &= \sum_{n=-\infty}^{\infty} \mathcal{O}_n z^{-n-h} \\ \mathcal{O}_n &= \oint \frac{dz}{2\pi i} z^{n+h-1}\mathcal{O}(z) \end{aligned} \tag{A.34}$$

$$\begin{aligned} &\mathcal{O}_n \left|1\right\rangle = 0 \qquad\quad \text{for } n \geq 1-h \\ &\mathcal{O}(0) \left|1\right\rangle = \mathcal{O}_{-h} \left|1\right\rangle \end{aligned} \tag{A.35}$$

$$\begin{aligned} h(c) &= -1 \\ h(b) &= 2 \\ h(\beta) &= 3/2 \\ h(\gamma) &= -1/2 \\ h\Big(e^{\ell\phi}\Big) &= -\frac{1}{2}\ell^2 - \ell \end{aligned} \tag{A.36}$$

$$\begin{aligned} c_n \left|0\right\rangle &= 0 \qquad\quad \text{for } n \geq 2 \\ b_n \left|0\right\rangle &= 0 \qquad\quad \text{for } n \geq -1 \\ \beta_r \left|0\right\rangle &= 0 \qquad\quad \text{for } r \geq -1/2 \\ \gamma_r \left|0\right\rangle &= 0 \qquad\quad \text{for } r \geq +3/2 \end{aligned} \tag{A.37}$$

$$\begin{aligned} b(z)\, c(0) &\sim \frac{1}{z} \\ \beta(z)\, \gamma(0) &\sim \frac{1}{z} \end{aligned} \tag{A.38}$$

$$\begin{aligned}
\eta(z)\,\xi(0) &\sim \frac{1}{z} \\
\chi(z)\,\chi(0) &\sim \ln z \\
\sigma(z)\,\sigma(0) &\sim \ln z \\
\phi(z)\,\phi(0) &\sim -\ln z
\end{aligned} \tag{A.39}$$

$$\gamma(z)\,\xi(0) \sim \frac{e^{\phi(0)}}{z} \tag{A.40}$$

$$\begin{aligned}
\delta(c) &= c \\
\delta(\gamma) &= e^{-\phi} \\
\delta(\beta) &= e^{+\phi}
\end{aligned} \tag{A.41}$$

shifted Bose-sea level:

$$|q\rangle := e^{q\,\phi}(0)\,|0\rangle \tag{A.42}$$

$$\begin{aligned}
\beta_r\,|q\rangle &= 0 \qquad r \geq -q-1/2 \\
\gamma_r\,|q\rangle &= 0 \qquad r \geq q+3/2
\end{aligned} \tag{A.43}$$

$$\begin{aligned}
\beta_r\,|q\rangle &= \oint \frac{dz}{2\pi i} z^{r+1/2} \beta(z)\, e^{q\phi}(0)\,|0\rangle \\
&= \oint \frac{dz}{2\pi i} z^{r+1/2} e^{-\phi} \partial\xi(z)\, e^{q\phi}(0)\,|0\rangle \\
&= \oint \frac{dz}{2\pi i} z^{r+1/2+q} \partial\xi(z) : e^{-\phi}(z)\, e^{q\phi}(0) : |0\rangle \\
&= 0 \qquad \text{for } r \geq -q-1/2
\end{aligned} \tag{A.44}$$

$|q\rangle$ is not $\mathrm{SL}(2,\mathbb{C})$-inavriant:

$$T^\phi = -\frac{1}{2}(\partial\phi)^2 - \partial^2\phi \tag{A.45}$$

$$\begin{aligned}
L_n^\phi\,|q\rangle &= 0 \qquad \text{for } n \geq 1 \\
L_0^\phi\,|q\rangle &= \left(-\frac{1}{2}q^2 - q\right)|q\rangle \qquad (= 0 \text{ for } q = 0, -2) \\
L_-1^\phi\,|q\rangle &\neq 0
\end{aligned} \tag{A.46}$$

$$
\begin{aligned}
|0\rangle_{\mathrm{NS}} &= \delta(\gamma(0))\,|0\rangle \;=\; e^{-\phi}(0)\,|0\rangle \\
|0\rangle_{\mathrm{R}} &= \Sigma(0)\,|0\rangle \;=\; e^{-\phi/2}(0)\,|0\rangle
\end{aligned}
\tag{A.47}
$$

$$
\begin{aligned}
Q_{\mathrm{B}} &= \oint \frac{dz}{2\pi i}\left(c\,T_{\mathrm{B}}^{\mathrm{m+g}} - bc\partial c\right) \\
&\quad + \oint \frac{dz}{2\pi i}\gamma\, T_{\mathrm{F}}^{\mathrm{m}} \\
&\quad + \oint \frac{dz}{2\pi i}\frac{1}{4}\gamma^2 b
\end{aligned}
\tag{A.48}
$$

$$
\begin{aligned}
j_{\mathrm{B}}(z) &= c\left(T^{\mathrm{m}} + \frac{1}{2}T^{\mathrm{g}}\right) + \gamma\left(G^{\mathrm{m}} + \frac{1}{2}G^{\mathrm{g}}\right) \\
&= c\left(T^{\mathrm{m}} + T^{\eta\xi} + T^{\phi}\right) + \eta e^{\phi}G^{\mathrm{m}} + bc\partial c - \eta\partial\eta e^{2\phi}b
\end{aligned}
\tag{A.49}
$$

$$
\begin{aligned}
T^{\eta\xi} &= (\partial\xi)\eta \\
T^{\phi} &= -\frac{1}{2}\partial\phi\partial\phi - \partial^2\phi
\end{aligned}
\tag{A.50}
$$

picture changin (raising) operator:

$$
\begin{aligned}
\mathrm{PCO}(z) &= Q\cdot\xi(z) = \{Q, \xi(z)\} \\
&= e^{\phi}G^{\mathrm{m}} + c\partial\xi + e^{2\phi}b\partial\eta + \partial(e^{2\phi}b\eta)
\end{aligned}
\tag{A.51}
$$

(*not* BRST-exact in *small* Hilbert space)

(inverse picture changin operator: $Y = c(\partial\xi)e^{-2\phi}$)

crucial rule: β, γ path integral is equal to the ϕ, η, ξ path integral with one additional ξ insertion and $\langle\xi(z)\rangle = 1$

decoupling of null states:

$$
\begin{aligned}
\langle\cdots\xi(w)\cdots[Q_{\mathrm{B}}, A(z)]\rangle \quad &\propto \quad \langle\cdots Q_{\mathrm{B}}\cdot\xi(w)\cdots A(z)\rangle \\
&= \quad \langle\cdots\mathrm{PCO}(w)\cdots A(z)\rangle \\
&\overset{\text{no }\xi\text{ mode left}}{=} \quad 0
\end{aligned}
\tag{A.52}
$$

position independence of PCO:

$$
\langle\mathrm{PCO}(z)\cdots\xi(w)\rangle = \langle\xi(z)\cdots\mathrm{PCO}(w)\rangle \tag{A.53}
$$

-1 picture: $\mathcal{V}^{-1} = e^{-\phi}\mathbf{\Psi}|_{\theta=0} = e^{-\phi}\mathcal{O}$

0 picture: $\mathcal{V}^0 = G_{-1/2}\cdot\mathcal{O}$

$$
\lim_{z\to 0} e^{\phi}T_{\mathrm{F}}^{\mathrm{m}}(z)\,e^{-\phi}\mathcal{O}(0) = \lim_{z\to 0} z \quad \underbrace{T_{\mathrm{F}}^{\mathrm{m}}(z)\,\mathcal{O}(0)}_{=G_{-1/2}\mathcal{O}/z+\text{order }z} \quad = \; \mathcal{V}^0 \tag{A.54}
$$

B. String fields and the star product

B.1 Star product in CFT language

LAB: String fields and the star product
CIT: Ohmori:2001,KishimotoOl

The CFT language formulation of the star product is nicely reviewed in [1, 3].

The integral $\int \Phi_1 * \cdots * \Phi_n$ is supposed to give the interaction vertex of n string fields Φ_n. Each of these can be thought of as propagating freely from $\tau_i = -\infty$ to $\tau_i = 0$, where they interact by pairwise overlap. We need a map that sends the n upper half planes in which the single strings propagate to a disk diagram in which the interaction takes place.

To that end, one first notes that the map

$$h(z) := \frac{1+iz}{1-iz} \tag{B.1}$$

- maps the upper half plane ($\mathrm{Im}(z) \geq 0$) bijectively to the unit disk $|h(z)| \leq 1$:

$$|h(z)|^2 = \frac{1+|z^2| - 2\mathrm{Im}(z)}{1+|z^2|+2\mathrm{Im}(z)} \leq 1\,, \tag{B.2}$$

$$h^{-1}(z) = -i\frac{z-1}{z+1}\,. \tag{B.3}$$

- maps the open string boundary (the real line) to the boundary of the unit disk:

$$|h(z)|^2 = 1 \Leftrightarrow \mathrm{Im}(z) = 0\,, \tag{B.4}$$

- maps the interaction point common to all three strings to the origin:

$$h(i) = 0\,. \tag{B.5}$$

But it also maps the upper unit half circle (the single string at $\tau_i = 0$) to the imaginary interval $i(-1,1)$:

$$h^*\left(e^{i\phi}\right) = \frac{1-e^{-i\phi}}{1+ie^{-i\phi}} = -h\left(e^{i\phi}\right)\,. \tag{B.6}$$

This means that the upper unit half disk is mapped to the *right* unit half disk (by a 90° rotation and a distortion). But in order to construct the trivalent vertex we want the upper unit half disk of the three strings to be mapped to three 120° wedges of the unit disk. (The desired cyclic invariance of the star product fixes the angles to 120°.) This is accomplished by shrinking the angle of the right unit disk ($h(z) \to h(z)^{2/3}$) and rotating the result appropriately by multiplying with a phase. It follows that the three holomorphic functions which map the three incoming strings to three segments of the unit disk are:

$$g_i^{(n)}(z) := e^{\frac{2\pi i}{n}(k-2)} h(z)^{2/n}\,. \tag{B.7}$$

LAB: def g in

This means that the integral of the triple star product is defined in terms of the correlator $\langle\cdots\rangle_{\mathrm{disk}}$ on the unit disk as

$$\int \Phi_1 * \Phi_2 * \cdots * \Phi_n = \left\langle (g_1^{(n)} \circ \Phi_1)(0)\,(g_2^{(n)} \circ \Phi_2)(0) \cdots (g_n^{(n)} \circ \Phi_n)(0) \right\rangle_{\mathrm{disk}}\,. \tag{B.8}$$

We may map the unit disk back to the upper half plane with h^{-1}. This defines the functions

$$f_i^{(n)}(z) := h^{-1}(g_i(z)) \tag{B.9}$$

and allows to write

$$\int \Phi_1 * \Phi_2 * \cdots \Phi_n = \left\langle (f_1^{(n)} \circ \Phi_1)(0)\,(f_2^{(n)} \circ \Phi_2)(0) \cdots (f_n^{(n)} \circ \Phi_n)(0) \right\rangle_{\rm UHP} . \tag{B.10}$$

In particular for the 2-point vertex the inversion map $I(z)$ turns up:

$$I(z) := f_1^{(2)}(z) = h^{-1}(-h(z)) = -\frac{1}{z} . \tag{B.11}$$

LAB: inversion map

For some applications it is furthermore useful to note that also

$$\begin{aligned}
(f_3^{(3)})^{-1}\left(f_2^{(3)}\right)(z) &= h^{-1}\left(\left(e^{-\frac{2\pi i}{3}} h\left(h^{-1}\left(h(z)^{2/3}\right)\right)\right)^{3/2}\right) \\
&= h^{-1}\left(\left(e^{-\frac{2\pi i}{3}} h(z)^{2/3}\right)^{3/2}\right) \\
&= h^{-1}(-h(z)) \\
&\overset{(B.11)}{=} -\frac{1}{z}
\end{aligned} \tag{B.12}$$

LAB: f3inv o f2

equals the inversion map.

B.2 The identity string field

Modulo some technical details there is a string field $\mathcal{I}$ which acts as the identity with respect to the star product, i.e. which satisfies

$$\mathcal{I} * A = A * \mathcal{I} = A \tag{B.13}$$

LAB: identity defintion

for a large class of string fields A. One observes that this implies in particular that

$$\begin{aligned}
\langle (f_2^{(2)} \circ \mathcal{I})(0)\,(f_2^{(2)} \circ \Phi)(0) \rangle &= \int \mathcal{I} * \Phi \\
&= \int \Phi \\
&= \langle (f_1^{(1)} \circ \Phi)(0) \rangle .
\end{aligned} \tag{B.14}$$

LAB: clean def of identity

In [3] it is argued that the relation

CIT: Kishimoto-toOhmori:2002

$$\langle (f_1^{(2)} \circ \mathcal{I})(0)\,(f_2^{(2)} \circ \Phi)(0) \rangle = \langle (f_1^{(1)} \circ \Phi)(0) \rangle_{\rm UHP} \tag{B.15}$$

is the cleanest way to *define* the identity element. It can be shown that this property implies (B.13) for states which, as Fock states, are normalizable, while it may fail on more general states.

One notes that due to $g_1^{(1)}(z) = h(z)^2$ the identifiy state maps corresponds to disk diagram where the upper half unit disk of a single string is mapped to the entire unit disk, with the right and the left half of the string overlapping. As a straightforward generalization one can define the *wedge states* (n) by

$$\int (n) * \mathcal{O} = \left\langle \left(f_2^{(n)} \circ \mathcal{O} \right)(0) \right\rangle_{\mathrm{UHP}} \tag{B.16}$$

where, according to (B.7),

$$f_2^{(n)} = h^{-1}\left(h(z)^{\frac{2}{n}} \right) \tag{B.17}$$

C. Boundary deformation theory

This section summarizes some aspects of boundary conformal field theory as discussed for instance in [13, 35, 36].

CIT: Recknagel Schomerus:1999,Schomerus:2002,

C.1 BCFTs

Given a conformal field theory on the complex plane (with coordinates $z, \bar z$) we get an associated ('descendant') *boundary conformal field theory* (BCFT) on the upper half plane (UHP), $\mathrm{Im}(z) > 0$, by demanding suitable boundary condition on the real line. The only class of cases well understood so far is that where the chiral fields $W(z)\,, \bar W(\bar z)$ can be analytically continued to the real line $\mathrm{Im}(z) = 0$ and a local automorphism of the chiral algebra exists, the *gluing map* Ω, such that on the boundary the left- and right-moving fields are related by

$$W(z) = \Omega \bar W(\bar z)\,, \qquad \text{at } z = \bar z\,. \tag{C.1}$$

LAB: gluing condition

In particular Ω always acts trivially on the energy momentum current

$$\Omega \bar T(\bar z) = \bar T(\bar z) \tag{C.2}$$

so that

$$T(z) = \bar T(\bar z)\,, \qquad \text{at } z = \bar z\,, \tag{C.3}$$

which ensures that no energy-momentum flows off the boundary.

This condition allows to introduce for every chiral $W, \bar W$ the single chiral field

$$\mathrm{W}(z) = \begin{cases} W(z) & \text{for } \mathrm{Im}(z) \geq 0 \\ \Omega\big(\bar W\big)(\bar z) & \text{for } \mathrm{Im}(z) < 0 \end{cases} \tag{C.4}$$

defined in the entire plane. (This is known as the 'doubling trick'.)

C.2 Boundary states

LAB: boundary states

Since it is relatively awkward to work with explicit constraints it is desirable to find a framework where the boundary condition on fields at the real line can be replaced by an operator insertion in a bulk theory without boundary.

Imagine an open string propagating with both ends attached to some D-brane. The worldsheet is topologically the disk (with appropriate operator insertions at the boundary). This disk can equivalently be regarded as the half sphere glued to the brane. But from this point of view it represents the worldsheet of a closed string with a certain source at the brane. Therefore the open string disk correlator on the brane is physically the same as a closed string emission from the brane with a certain source term corresponding to the open string boundary condition. The source term at the boundary of the half sphere can be represented by an operator insertion in the full sphere. The state corresponding to this vertex insertion is the *boundary state*.

In formal terms this heuristic picture translates to the following procedure:

First map the open string worldsheet to the sphere, in the above sense. By stereographic projection, the sphere is mapped to the plane and the upper half sphere which represents the open string worldsheet disk gets mapped to the complement of the unit disk in the plane. Denote the complex coordinates on this complement by $\zeta, \bar{\zeta}$ and let the open string worldsheet time $\tau = -\infty$ be mappped to $\zeta = 1$ and $\tau = +\infty$ mapped to $\zeta = -1$ (so that the open string propagates 'from right to left' in these worldsheet coordinates). With $z, \bar{z}$ the coordinates on the UHP this corresponds to $z = 0 \mapsto \zeta = 1$ and $z = \infty \mapsto \zeta = -1$. The rest of the boundary of the string must get mapped to the unit circle, which is where the string is glued to the brane. An invertible holomorphic map from the UHP to the complement of the unit disk with these features[7] is

$$\zeta(z) := \frac{1 - iz}{1 + iz}\,. \tag{C.6}$$

For a given boundary condition α the boundary state $|\alpha\rangle$ is now defined as the state corresponding to the operator which, when inserted in the sphere, makes the correlator of some open string field Φ on the sphere equal to that on the UHP with boundary condition α:

$$\langle \Phi^{(\mathrm{H})}(z,\bar{z})\rangle_\alpha = \left(\frac{\partial\zeta}{\partial z}\right)^h \left(\frac{\partial\bar{\zeta}}{\partial\bar{z}}\right)^{\bar{h}} \langle 0|\, \Phi^{(\mathrm{P})}\big(\zeta,\bar{\zeta}\big)\, |\alpha\rangle\ . \tag{C.7}$$

Noting that on the boundary we have

$$\frac{\partial\zeta}{\partial z} = -i\zeta\,, \qquad \text{at } z = \bar{z} \Leftrightarrow \zeta = 1/\bar{\zeta} \tag{C.8}$$

the gluing condition (C.1) becomes in the new coordinates

$$\begin{aligned} \left(\frac{\partial\zeta}{\partial z}\right)^h W(\zeta) &= \left(\frac{\partial\bar{\zeta}}{\partial\bar{z}}\right)^h \Omega\bar{W}\big(\bar{\zeta}\big) \\ \Leftrightarrow W(\zeta) &= (-1)^h \bar{\zeta}^{2h}\, \Omega\bar{W}\big(\bar{\zeta}\big)\ , \qquad \text{at } \zeta = 1/\bar{\zeta}\,. \end{aligned} \tag{C.9}$$

LAB: gluing condition in z

[7]

$$|\zeta|^2 = \frac{1 + |z|^2 + 2\mathrm{Im}(z)}{1 + |z|^2 - 2\mathrm{Im}(z)} \geq 0 \quad \text{for } \mathrm{Im}(z) \geq 0 \tag{C.5}$$

In the theory living on the plane this condition translates into a constraint on the boundary state $|\alpha\rangle$:

$$\begin{aligned} 0 &\stackrel{!}{=} \langle 0| \cdots \sum_{n=-\infty}^{\infty} \left(W_n \zeta^{-n-h} - (-1)^h \zeta^{-2h} \Omega \bar{W}_n \zeta^{n+h} \right) |\alpha\rangle \\ &= \langle 0| \cdots \sum_{n=-\infty}^{\infty} \left(W_n \zeta^{-n-h} - (-1)^h \Omega \bar{W}_n \zeta^{n-h} \right) |\alpha\rangle \,, \qquad \forall\, \zeta = 1/\bar{\zeta}\,, \end{aligned} \tag{C.10}$$

i.e.

$$\left(W_n - (-1)^h \Omega \bar{W}_{-n} \right) |\alpha\rangle = 0\,, \qquad \forall\, n \in \mathbb{N}\,. \tag{C.11}$$

LAB: constraint on bound

Since $\Omega\bar{T} = \bar{T}$ holds for all BCFTs this implies in particular that one always has

$$\left(L_n - \bar{L}_{-n} \right) |\alpha\rangle = 0 \qquad \forall\, n\,, \tag{C.12}$$

LAB: Virasoro constraint

which says that $|\alpha\rangle$ is invariant with respect to reparameterizations of the spatial worldsheet variable σ parameterizing the boundary (*cf.* for instance section 3 of [14]).

CIT: Schreiber:2004

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