

# Nonabelian 2-forms and loop space connections from SCFT deformations

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ABSTRACT: It is shown how the deformation of the superconformal generators on the string's worldsheet by a nonabelian super-Wilson line gives rise to a covariant exterior derivative on loop space coming from a nonabelian 2-form on target space. The expression obtained this way is new in the context of strings (but has been considered before in the context of integrable systems), and its consistency is verified by checking that its global gauge transformations on loop space imply the familiar gauge transformations on target space. We derive the second order gauge transformation from infinitesimal local gauge transformations on loop space and find that a consistent picture is obtained only when the sum of the 2-form and the 1-form field strengths vanish. The same condition has recently been derived from 2-group gauge theory reasoning, where it appears to be an obstacle. We observe that this condition implies that the connection on loop space is *flat*. Finally we compute the background equations of motion of the nonabelian 2-form by canceling divergences in the deformed boundary state.

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## 1. Introduction

The target space theories which give rise to non-abelian 2-forms are not at all well understood [1]. One expects [2, 3] that they involve stacks of 5-branes on which open membranes may end [4, 5, 6]. This has recently been made more precise [7] using anomaly cancellation on M5-branes and the language of nonabelian gerbes developed in [8]. The boundary of these membranes appear as strings, [9, 10], (self-dual strings [11, 12, 13], “little strings” [14], fundamental strings or D-strings [15]) in the world-volume theory of the 5-branes [6], generalizing [1] the way how open string endpoints appear as “quarks” in the world-volume theory of D-branes. Just like a nonabelian 1-form couples to these “quarks”, i.e. to the boundary of an open string, a (possibly non-abelian) 2-form should couple [16] to the boundary of an open membrane [10, 17, 18, 19], i.e. a to string on the (stack of) 5 branes. One proposal for how such a non-abelian  $B$  field might be induced by a stack of branes has been made in [16]. A more formal derivation of the non-abelian 2-forms arising on stacks of M5 branes is given in [7]. General investigations into the possible nature of such non-abelian 2-forms have been done for instance in [20, 21].

(From the point of view of the effective 6-dimensional supersymmetric worldvolume theory of the 5-branes these 2-form field(s) come either from a tensor multiplet or from a gravitational multiplet of the worldvolume supersymmetry representation [22].)

This analogy strongly suggests that there is a *single* Chan-Paton-like factor associated to each string living on the stack of 5 branes, indicating which of the  $N$  branes in the stack it is associated with. This Chan-Paton factor should be the degree of freedom that the non-abelian  $B$ -field acts on.

Hence the higher-dimensional generalization of ordinary gauge theory should, in terms of strings, involve the steps upwards the dimensional ladder indicated in table 1.

(1-)gauge theory	2-gauge theory
string ending on D-brane	→ membrane ending on NS brane
“quark” on D-brane	→ string on NS brane
nonabelian 1-form gauge field	→ nonabelian 2-form gauge field $B$
coupling to the boundary of a 1-brane (string)	→ coupling to the boundary of a 2-brane (membrane)
Chan-Paton factor indicating which D-brane in the stack the “quark” sits on	→ Chan-Paton-like factor indicating with NS brane in the stack the membrane boundary string sits on.

**Table 1:** Expected relation between 1-form and 2-form gauge theory in stringy terms

These considerations receive substantiation by the fact that indeed the contexts in which nonabelian 2-forms have been argued to arise naturally are the worldsheet theories on these NS 5-branes [3, 2, 9, 15, 1, 23].

The study of little strings, tensionless strings and  $N = (2, 0)$  QFTs in six dimensions is involved and no good understanding of any non-abelian 2-form from this target space perspective has emerged so far. However, a compelling connection is the relation of these 6-dimensional theories, upon compactification, to Yang-Mills theory in 4-dimensions, where

the 1-form gauge field of the Yang-Mills theory arises as one component of the 2-form in the 6-dimensional theory [23].

For this to work the dimension  $d = 5 + 1$  of the world-volume theory of the 5-branes plays a crucial role, because here the 2-form  $B$  can have and does have *self-dual* field strength  $H = \star H$  [10, 23] (related to the existence of the self-dual strings in 6 dimensions first discussed in [24]).

But this means that there cannot be any ordinary non-topological action of the form  $dH \wedge \star dH$  for the  $B$ -field, and that furthermore the dynamical content of the  $B$  field would essentially be that of a 1-form  $\alpha$  [23]: Namely when the 1+5 dimensional field theory is compactified on a circle and  $B$  is rewritten as

$$B = B_{ij} dx^i \wedge dx^j + \alpha_i dx^i \wedge dx^6 \quad \text{for } i, j \in \{1, 2, 3, 4, 5\}.$$

with  $\partial_6 B = 0$ , then  $dB = \star dB$  implies that in five dimensions  $B$  is just dual to  $\alpha$

$$d^{(5)}B = \star^{(5)}d\alpha. \tag{1.1}$$

In particular, since the compactified theory should give possibly non-abelian Yang-Mills with  $\alpha$  the gauge field [23] it is natural to expect [3] that in the uncompactified theory there must be a non-abelian  $B$  field. Since there is no Lagrangian description of the brane's worldvolume theory [25, 2] it is hard to make this explicit. This is one reason why it seems helpful to consider the worldsheet theory of strings propagating in the 6-dimensional brane volume, as we will do here. The non-abelian Yang-Mills theory in the context of NS 5-branes considered in [9] uses  $n$  D4-branes suspended between two NS5-branes. The former can however be regarded as a single M5-brane wrapped  $n$  times around the  $S^1$  (*cf.* p. 34 of [9]).

In [26] it is argued that, while the worldvolume theory on a stack of 5-branes with non-abelian 2-form fields is not known, it cannot be a local field theory. This harmonizes with the attempts in [1] to define it in terms of “nonabelian surface equations” which are supposed to generalize the well-known Wilson loop equations of ordinary Yang-Mills theory to Wilson *surfaces*. These Wilson surfaces become ordinary Wilson loops in loop space, and these play a pivotal role in the constructions presented here.

In the following we shall make no attempt to say anything *directly* about the physics of membranes attached to 5 branes. Instead the strategy here is to look at the worldsheet theory of strings and to see from the worldsheet perspective if anything can be said about superconformal field theories that involve a non-abelian 2-form. Even though we will not try to exhibit a direct correspondence between certain such SCFTs and the target space physics of membrane boundary strings on NS branes, we will be able to recognize in the formal structure of these SCFTs some of the above mentioned expected properties of such theories. In the process we will clarify or shed new light on previous approaches to non-abelian surface holonomy [3, 27], in particular by deriving the form of the loop space connection from SCFT deformations (boundary state deformations) and deriving its crucial flatness condition from consistency conditions on its gauge invariance. This flatness condition turns out to be already known [28] in the context of 2-group theory [29], and

together with the form of the loop space connection used here it is seen to solve the famous old problem noted [30], related to the construction of a sensible notion of surface ordering for non-abelian 2-forms. By merging the insights into non-abelian surface holonomy using the loop space results found here and in [27] with those of 2-group theory, one obtains a coherent picture clarifying aspects of both methods. This is reported in full detail in [31].

The aim of this paper is to show that it is possible to learn about the physics of strings in non-abelian 2-form backgrounds by suitably generalizing known deformation techniques of 2d superconformal field theories from the abelian to the nonabelian case, this way learning about the nonabelian target space background and its effective nonabelian 2-form field theory from the study of appropriate worldsheet theories.

In particular, the technique of SCFT deformations using ‘‘Morse theory methods’’ in loop space formalism [32] that was studied in [33], together with insights into boundary state deformations obtained in [34, 35], is used to construct superconformal worldsheet generators that incorporate a connection on loop space induced by a nonabelian 2-form on target space.

Given a nonabelian 1-form  $A$  and a 2-form  $B$  on target space (which we think of as taking values in some matrix algebra) we argue from general SCFT deformation theory that the connection on loop space (the space of maps from the circle into target space) induced by this background can be read off from a similarity transformation of the worldsheet supercharges  $G$  and  $\bar{G}$  by the operator

$$\exp(\mathbf{W})^{(A)(B)_{\text{nonab}}} = \text{P exp} \left( \int_0^{2\pi} d\sigma \left( iA_\mu X'^\mu + \frac{1}{2} \left( \frac{1}{T} F_A + B \right)_{\mu\nu} \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu} \right) \right). \quad (1.2)$$

Here  $X(\sigma)$  is the map from the loop into target space,  $X'(\sigma) = \frac{d}{d\sigma} X(\sigma)$  is the tangent vector,  $T = 1/2\pi\alpha'$  is the string tension,  $F_A$  the field strength of  $A$  and  $\mathcal{E}^\dagger$  are operators of exterior multiplication with differential forms on loop space, or, equivalently, linear combinations of worldsheet fermions. P denotes path ordering.

The resulting gauge covariant exterior derivative  $\mathbf{d}^{(A)(B)}$  on loop space will be shown to be

$$\mathbf{d}^{(A)(B)} = \mathbf{d} + iT \int_0^{2\pi} d\sigma U_A(2\pi, \sigma) B_{\mu\nu} X'^\mu \mathcal{E}^{\dagger\nu}(\sigma) U_A(\sigma, 2\pi), \quad (1.3)$$

where  $\mathbf{d}$  is the ordinary exterior derivative on loop space,  $U_A$  is the holonomy of  $A$  along the loop and  $U_A(\sigma, \kappa) = U_A^\dagger(\kappa, \sigma)$ . Note that this connection (1.3) is indeed a 1-form on loop space, taking values in the respective nonabelian algebra carried by  $B$  and  $A$ .

This result is similar to, but slightly and crucially different from, the construction proposed in [3], the difference being the second  $U_A$  factor on the right. It agrees however with the form of the loop space connection used in [27] in the context of integrable systems as well as with that proposed in [36] in the context of non-abelian gerbes.

As a first consistency check, *global* gauge transformations of (1.3) can be seen to reproduce the usual target space gauge symmetry  $A \mapsto UAU^\dagger + U(dU^\dagger)$ ,  $B \mapsto UBU^\dagger$ .

The fact that, as we shall discuss, the operator (1.2) also serves as a deformation operator for boundary states, will be shown to make it quite transparent that *local* infinitesimal gauge transformations on loop space make sense only when the fermionic terms in (1.2) vanish. This is the case precisely when the 1-form field strength  $F_A$  cancels the 2-form field:

$$\frac{1}{T}F_A + B = 0. \quad (1.4)$$

We will check that in this case the connection (1.3) is *flat*. This is an important sufficient condition for the connection on loop space to assign well-defined surface holonomy independent of foliation of that surface by loops.

In fact, the condition (1.4) had been derived recently [28] from consistency conditions in 2-group theory [29] (that are missing in the otherwise similar approach [36]). From the point of view of some approaches to higher gauge theory it may seem like an obstacle for writing down interesting Lagrangians for theories. From our point of view it seems however to be the necessary condition for a consistent coupling of the string to the nonabelian background. More discussion of this point is given in §3.5 (p.22) and a detailed analysis of nonabelian loop space connections, their relation to 2-group theory and the conditions on non-abelian 2-forms that they imply will be presented in [31].

In order to further clarify this point, we compute the equations of motion of the nonabelian background by, following [34, 35] (the relation to the alternative approach [37] will be discussed in [38]), canceling divergences that appear when acting with (1.2) on the boundary state of a bare brane. The resulting equations of motion are, to lowest order,

$$\operatorname{div}_A B = 0 = \operatorname{div}_A F_A, \quad (1.5)$$

where  $\operatorname{div}_A$  is the gauge-covariant divergence. In terms of the gauge field this are just the Yang-Mills equations. Some aspects of this result will be discussed.

The structure of this paper is as follows:

In §2 (p.6) the SCFT deformation technique and loop space formalism studies in [33] is reviewed and some new aspects like nonabelian differential forms on loop space as well as gauge connections on loop space are discussed.

§3 (p.13) applies these techniques to a certain non-abelian generalization of the previously studied abelian case and this way identifies a nonabelian connection on loop space, coming from a 2-form on target space, as part of a superconformal algebra which should describe strings in nonabelian 2-form backgrounds.

Some concluding remarks are given in §4 (p.26). The appendix §A (p.28) reviews some aspects of boundary state formalism that are referred to in the main text. Appendix §B (p.30) gives some calculation omitted from the main text.

## 2. SCFT deformations in loop space formalism

This introductory section discusses aspects of loop space formalism and deformation theory that will be applied in §3 (p.13) to the description of nonabelian 2-form background fields.

### 2.1 SCFT deformations and backgrounds using Morse theory technique

The reasoning by which we intend to derive the worldsheet theory for superstrings in non-abelian 2-form backgrounds involves an interplay of deformation theory of superconformal field theories for closed strings, as described in [33], as well as the generalization to boundary state deformations, which are discussed further below in §2.2 (p.10). The deformation method we use consists of adding deformation terms to the super Virasoro generators and in this respect is in the tradition of similar approaches as for instance described in [39, 40, 41, 42, 43] (as opposed to, say, deformations of the CFT correlators). What is new here is the systematic use of similarity transformations on a certain combination of the supercharges, as explained below.

In this section the SCFT deformation technique for the closed string is briefly reviewed in a manner which should alleviate the change of perspective from the string's Fock space to loop space.

Consider some realization of the superconformal generators  $L_n, \bar{L}_n, G_r, \bar{G}_r$  (we follow the standard notation of [44]) of the type II superstring. We are looking for consistent deformations of these operators to operators  $L_n^\Phi, \bar{L}_n^\Phi, G_r^\Phi, \bar{G}_r^\Phi$  ( $\Phi$  indicates some unspecified background field configuration which is associated with the deformation) which still satisfy the superconformal algebra and so that the generator of spatial worldsheet reparametrizations remains invariant:

$$L_n^\Phi - \bar{L}_{-n}^\Phi \stackrel{!}{=} L_n - \bar{L}_{-n}. \quad (2.1)$$

This condition follows from a canonical analysis of the worldsheet action, which is nothing but 1+1 dimensional supergravity coupled to various matter fields. As for all gravitational theories, their ADM constraints break up into spatial diffeomorphism constraints as well as the Hamiltonian constraint, which alone encodes the dynamics.

The condition (2.1) can also be understood in terms of boundary state formalism, which is briefly reviewed in §A (p.28). As discussed below, the operator  $\mathcal{B}$  related to a nontrivial boundary state  $|\mathcal{B}\rangle$  can be interpreted as inducing a deformation  $G_r^\Phi := \mathcal{B}^{-1}G_r\mathcal{B}$ , etc. and the condition (2.1) is then equivalent to (A.12).

In any case, we are looking for isomorphisms of the superconformal algebra which satisfy (2.1):

To that end, let  $d_r$  and  $d_r^\dagger$  be the modes of the polar combinations of the left- and right-moving supercurrents

$$\begin{aligned} d_r &:= G_r + i\bar{G}_{-r} \\ d_r^\dagger &:= (d_r)^\dagger = G_r - i\bar{G}_{-r}. \end{aligned} \quad (2.2)$$

These are the 'square roots' of the reparametrization generator

$$\mathcal{L}_n := -i(L_n - \bar{L}_{-n}), \quad (2.3)$$

i.e.

$$\{d_r, d_s\} = \{d_r^\dagger, d_s^\dagger\} = 2i\mathcal{L}_{r+s}. \quad (2.4)$$

Under a deformation the right hand side of this equation must stay invariant (2.1) so that

$$\begin{aligned} d_r^\Phi &:= d_r + \Delta_\Phi d_r \\ d_r^{\dagger\Phi} &:= d_r^\dagger + (\Delta_\Phi d_r)^\dagger \end{aligned} \quad (2.5)$$

implies that the shift  $\Delta_\Phi d_r$  of  $d_r$  has to satisfy

$$\{d_r, \Delta_\Phi d_s\} + \{d_s, \Delta_\Phi d_r\} + \{\Delta_\Phi d_r, \Delta_\Phi d_s\} = 0. \quad (2.6)$$

One large class of solutions of this equation is

$$\Delta_\Phi d_r = A^{-1} [d_r, A], \quad \text{for } [\mathcal{L}_n, A] = 0 \quad \forall n, \quad (2.7)$$

where  $A$  is any even graded operator that is spatially reparametrization invariant, i.e. which commutes with (2.3).

When this is rewritten as

$$\begin{aligned} d_r^\Phi &= A^{-1} \circ d_r \circ A \\ d_r^{\dagger\Phi} &= A^\dagger \circ d_r^\dagger \circ A^{\dagger-1} \end{aligned} \quad (2.8)$$

one sees explicitly that the formal structure involved here is a direct generalization of that used in [45] in the study of the relation of deformed generators in supersymmetric quantum *mechanics* to Morse theory. Here we are concerned with the direct generalization of this mechanism from  $1+0$  to  $1+1$  dimensional supersymmetric field theory.

In  $1+0$  dimensional SQFT (i.e. supersymmetric quantum mechanics) relation (2.8) is sufficient for the deformation to be truly an isomorphism of the algebra of generators. In  $1+1$  dimensions, on the superstring's worldsheet, there is however one further necessary condition for this to be the case. Namely the (modes of the) new worldsheet Hamiltonian constraint  $H_n = L_n + \bar{L}_- n$  must clearly be defined as

$$H_n^\Phi := \frac{1}{2} \left\{ d_r^\Phi, d_{n-r}^{\dagger\Phi} \right\} - \delta_{n,0} \frac{c}{12} (4r^2 - 1) \quad (2.9)$$

and (2.6) alone does not guarantee that this is *unique* for all  $r \neq n/2$ . If it is, however, then the Jacobi identity already implies that

$$\begin{aligned} G_r^\Phi &:= \frac{1}{2} \left( d_r^\Phi + d_r^{\dagger\Phi} \right) \\ L_n^\Phi &:= \frac{1}{4} \left( \left\{ d_r^\Phi, d_{n-r}^{\dagger\Phi} \right\} + \left\{ d_r^\Phi, d_{n-r}^\Phi \right\} \right) - \delta_{r,n/2} \frac{c}{24} (4r^2 - 1) \\ \bar{G}_r^\Phi &:= -\frac{i}{2} \left( d_{-r}^\Phi - d_{-r}^{\dagger\Phi} \right) \\ L_n^\Phi &:= \frac{1}{2} \left( \left\{ d_{-r}^\Phi, d_{r-n}^{\dagger\Phi} \right\} - \left\{ d_{-r}^\Phi, d_{r-n}^\Phi \right\} \right), \quad \forall r \neq n/2 \end{aligned} \quad (2.10)$$

generate two mutually commuting copies of the super Virasoro algebra.

In order to see this first note that the two copies of the unperturbed Virasoro algebra in terms of the 'polar' generators  $d_r, d_r^\dagger, i\mathcal{L}_m, H_m$  read

$$\begin{aligned}
\{d_r, d_s\} &= 2i\mathcal{L}_{r+s} = \left\{d_r^\dagger, d_s^\dagger\right\} \\
[i\mathcal{L}_m, d_r] &= \frac{m-2r}{2}d_{m+r} \\
[i\mathcal{L}_m, d_r^\dagger] &= \frac{m-2r}{2}d_{m+r}^\dagger \\
[i\mathcal{L}_m, i\mathcal{L}_n] &= (m-n)i\mathcal{L}_{m+n} \\
[i\mathcal{L}_m, H_n] &= (m-n)i\mathcal{H}_{m+n} + \frac{c}{6}(m^3 - m)\delta_{m,-n} \\
[H_m, d_r] &= \frac{m-2r}{2}d_{m+r}^\dagger \\
[H_m, d_r^\dagger] &= \frac{m-2r}{2}d_{m+r} \\
[H_m, H_n] &= (m-n)i\mathcal{L}_{m+n}.
\end{aligned} \tag{2.11}$$

Now check that these relations are obeyed also by the deformed generators  $d_r^\Phi, d_r^{\dagger\Phi}, i\mathcal{L}_m, H_m^\Phi$  using the two conditions (2.8) and (2.9):

First of all the relations

$$\begin{aligned}
[i\mathcal{L}_m, d_r^\Phi] &= \frac{m-2r}{2}d_{m+r}^\Phi \\
[i\mathcal{L}_m, d_r^{\dagger\Phi}] &= \frac{m-2r}{2}d_{m+r}^{\dagger\Phi}
\end{aligned} \tag{2.12}$$

follow simply from (2.8) and the original bracket  $[L_m, G_r] = \frac{m-2r}{2}G_{m+r}$  and immediately imply

$$[i\mathcal{L}_m, i\mathcal{L}_n] = (m-n)i\mathcal{L}_{m+n} \tag{2.13}$$

(note that here the anomaly of the left-moving sector cancels that of the right-moving one).

Furthermore

$$\begin{aligned}
[i\mathcal{L}_m, H_n^\Phi] &= \left[ i\mathcal{L}_m, \frac{1}{2} \left\{ d_r^\Phi, d_{n-r}^{\dagger\Phi} \right\} \right] \\
&\stackrel{(2.12)}{=} \frac{m-2r}{4} \left\{ d_{m+r}^\Phi, d_{n-r}^{\dagger\Phi} \right\} + \frac{m-2(n-r)}{4} \left\{ d_r^\Phi, d_{m+n-r}^{\dagger\Phi} \right\} \\
&\stackrel{(2.9)}{=} (m-n)H_{m+n}^\Phi + \delta_{m,-n} \frac{c}{6} \left( \frac{m-2r}{4} (4(m+r)^2 - 1) + \frac{m-2(n-r)}{4} (4r^2 - 1) \right) \\
&= (m-n)H_{m+n}^\Phi + \delta_{m,-n} \frac{c}{6} (m^3 - m).
\end{aligned} \tag{2.14}$$

(Here the anomalies from both sectors add.)

The commutator of the Hamiltonian with the supercurrents is obtained for instance by first writing:

$$[H_m^\Phi, d_r^\Phi] = \frac{1}{2} \left[ \left\{ d_r^\Phi, d_{m-r}^{\dagger\Phi} \right\}, d_r^\Phi \right]$$

$$\begin{aligned}
&= -\frac{1}{2} \left[ \{d_r^\Phi, d_r^\Phi\}, d_{m-r}^{\dagger\Phi} \right] - \frac{1}{2} \left[ \{d_r^\Phi, d_{m-r}^{\dagger\Phi}\}, d_r^\Phi \right] \\
&= - \left[ i\mathcal{L}_{2r}, d_{m-r}^{\dagger\Phi} \right] - [H_m^\Phi, d_r^\Phi] \\
&= (m-2r)d_{m+r}^{\dagger\Phi} - [H_m^\Phi, d_r^\Phi], \tag{2.15}
\end{aligned}$$

from which it follows that

$$[H_m^\Phi, d_r^\Phi] = \frac{(m-2r)}{2} d_{m+r}^{\dagger\Phi} \tag{2.16}$$

and similarly

$$[H_m^\Phi, d_r^{\dagger\Phi}] = \frac{(m-2r)}{2} d_{m+r}^\Phi. \tag{2.17}$$

This can finally be used to obtain

$$[H_m^\Phi, H_n^\Phi] = (m-n)i\mathcal{L}_{m+n}. \tag{2.18}$$

In summary this shows that every operator  $A$  which

1. commutes with  $i\mathcal{L}_m$
2. is such that  $\left\{ A^{-1}d_r A, A^\dagger d_{n-r}^\dagger A^{\dagger-1} \right\} - \delta_{n,0} \frac{c}{12}(4r^2 - 1)$  is *independent* of  $r$

defines a consistent deformation of the super Virasoro generators and hence a string background which satisfies the classical equations of motion of string field theory.

In [33] it was shown how at least all massless NS and NS-NS backgrounds can be obtained by deformations  $A$  of the form  $A = e^{\mathbf{W}}$ , where  $\mathbf{W}$  is related to the vertex operator of the respective background field. For instance a Kalb-Ramond  $B$ -field background is induced by setting

$$\mathbf{W}^{(B)} = \frac{1}{2} \int d\sigma \left( \frac{1}{T} dA + B \right)_{\mu\nu} \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu}, \tag{2.19}$$

where  $\mathcal{E}^\dagger$  are operators of exterior multiplication with differential forms on loop space, to be discussed in more detail below in §2.3.1 (p.11), and we have included the well known contribution of the 1-form gauge field  $A$ .

Moreover, it was demonstrated in [46] that the structure (2.8) of the SCFT deformations allows to handle superstring evolution in nontrivial backgrounds as generalized Dirac-Kähler evolution in loop space.

In the special case where  $A$  is *unitary* the similarity transformations (2.8) of  $d$  and  $d^\dagger$  and hence of all other elements of the super-Virasoro algebra are identical and the deformation is nothing but a unitary transformation. It was discussed in [33] that gauge transformations of the background fields, such as reparameterizations or gauge shifts of the Kalb-Ramond field, are described by such unitary transformation.

In particular, an *abelian* gauge field background was shown to be induced by the Wilson line

$$\mathbf{W}^{(A)} = i \oint d\sigma A_\mu(X(\sigma)) X'^\mu(\sigma) \tag{2.20}$$

of the gauge field along the closed string.

While the above considerations apply to closed superstrings, in this paper we shall be concerned with open superstrings, since these carry the Chan-Paton factors that will transform under the nonabelian group that we are concerned with in the context on nonabelian 2-form background fields.

It turns out that the above method for obtaining closed string backgrounds by deformations of the differential geometry of loop space nicely generalizes to open strings when boundary state formalism is used. This is the content of the next section.

## 2.2 Boundary state deformations from unitary loop space deformations

The tree-level diagram of an open string attached to a D-brane is a disk attached to that brane with a certain boundary condition on the disk characterizing the presence of the D-brane. In what is essentially a generalization of the method of image charges in electrostatics this can be equivalently described by the original disk “attached” to an auxiliary disc, so that a sphere is formed, and with the auxiliary disk describing incoming closed strings in just such a way, that the correct boundary condition is reproduced.

Some details behind this heuristic picture are recalled in §A (p.28). For our purposes it suffices to note that a deformation (2.8) of the superconformal generators for closed strings with  $A$  a *unitary* operator (as for instance given by (2.20)) is of course equivalent to a corresponding unitary transformation of the closed string states. But this means that the boundary state formalism implies that open string dynamics in a given background described by a unitary deformation operator  $A$  on loop space is described by a boundary state  $A^\dagger |D9\rangle$ , where  $|D9\rangle$  is the boundary state of a bare space-filling brane, which again, as discussed below in (2.30), is nothing but the constant 0-form on loop space.

In this way boundary state formalism rather nicely generalizes the loop space formalism used here from closed to open strings.

In a completely different context, the above general picture has in fact been verified for abelian gauge fields in [34, 35]. There it is shown that acting with (2.20) and the unitary part<sup>1</sup> of (2.19) for  $B = 0$  on  $|D9\rangle$ , one obtains the correct boundary state deformation operator

$$\exp\left(\mathbf{W}^{(A)(B=\frac{1}{T}dA)}\right) \exp\left(\int_0^{2\pi} \left(iA_\mu X'^\mu + \frac{1}{2T}(dA)_{\mu\nu} \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu}\right)\right) \quad (2.21)$$

which describes open strings on a D9 brane with the given gauge field turned on.

Here, we want to show how this construction directly generalizes to deformations describing nonabelian 1- and 2-form backgrounds. It turns out that the loop space perspective together with boundary state formalism allows to identify the relation between the nonabelian 2-form background and the corresponding connection on loop space, which again allows to get insight into the gauge invariances of gauge theories with nonabelian 2-forms.

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<sup>1</sup>When acting on  $|D9\rangle$  the non-unitary part of (2.19) is projected out automatically.

One simple observation of the abelian theory proves to be crucial for the non-abelian generalization: Since (2.21) *commutes* with  $\mathbf{d}_K$  the loop space connection it induces (following the reasoning to be described in §2.3.2 (p.13)) *vanishes*. This makes good sense, since the closed string does not feel the background  $A$  field.

But the generalization of a *vanishing* loop space connection to something less trivial but still trivial enough so that it can describe something which does not couple to the closed string is a *flat* loop space connection. Flatness in loop space means that every closed curve in loop space, which is a torus worldsheet (for the space of oriented loops) in target space, is assigned surface holonomy  $g = 1$ , the identity element. This means that only open worldsheets with boundary can feel the presence of a flat loop space connection, just as it should be.

From this heuristic picture we expect that abelian but flat loop space connections play a special role. Indeed, we shall find in §3.4 (p.20) that only these are apparently well behaved enough to avoid a couple of well known problems.

The next section first demonstrates that the meaning of the above constructions become rather transparent when the superconformal generators are identified as deformed deRham operators on loop space.

## 2.3 Superconformal generators as deformed deRham operators on loop space.

Details of the representation of the super Virasoro generators on loop space have been given in [33] and we here follow the notation introduced there.

### 2.3.1 Differential geometry on loop space

Again, the loop space formulation can nicely be motivated from boundary state formalism:

The boundary state  $|b\rangle$  describing the space-filling brane in Minkowski space is, according to (A.11), given by the constraints

$$\begin{aligned} (\alpha_n^\mu + \bar{\alpha}_{-n}^\mu) |b\rangle &= 0, & \forall n, \mu \\ (\psi_r^\mu - i\bar{\psi}_{-r}^\mu) |b\rangle &= 0, & \forall r, \mu \end{aligned} \quad (2.22)$$

(in the open string R sector).

We can think of the super-Virasoro constraints as a Dirac-Kähler system on the exterior bundle over loop space  $\mathcal{L}(\mathcal{M})$  with coordinates

$$X^{(\mu, \sigma)} = \frac{1}{\sqrt{2\pi}} X_0^\mu + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^\mu - \bar{\alpha}_{-n}^\mu) e^{in\sigma}, \quad (2.23)$$

holonomic vector fields

$$\frac{\delta}{\delta X^\mu(\sigma)} := \partial_{(\mu, \sigma)} = i\sqrt{\frac{T}{4\pi}} \sum_{n=-\infty}^{\infty} \eta_{\mu\nu} (\alpha_n^\nu + \bar{\alpha}_{-n}^\nu) e^{in\sigma} \quad (2.24)$$

differential form creators

$$\begin{aligned}\mathcal{E}^{\dagger(\mu,\sigma)} &= \frac{1}{2} (\psi_+^\mu(\sigma) + i\psi_-^\mu(\sigma)) \\ &= \frac{1}{\sqrt{2\pi}} \sum_r (\bar{\psi}_{-r} + i\psi_r) e^{ir\sigma}\end{aligned}\tag{2.25}$$

and annihilators

$$\begin{aligned}\mathcal{E}^{(\mu,\sigma)} &= \frac{1}{2} (\psi_+^\mu(\sigma) - i\psi_-^\mu(\sigma)) \\ &= \frac{1}{\sqrt{2\pi}} \sum_r (\bar{\psi}_{-r} - i\psi_r) e^{ir\sigma}.\end{aligned}\tag{2.26}$$

In the polar form (2.2) the fermionic super Virasoro constraints are identified with the modes of the exterior derivative on loop space

$$\mathbf{d}_K = \int_0^{2\pi} d\sigma \left( \mathcal{E}^{\dagger\mu} \partial_\mu(\sigma) + iTX'^\mu \mathcal{E}_\mu(\sigma) \right),\tag{2.27}$$

deformed by the reparametrization Killing vector

$$K^{(\mu,\sigma)} := X'^\mu(\sigma),\tag{2.28}$$

where  $T = \frac{1}{2\pi\alpha'}$  is the string tension. The Fourier modes of this operator are the polar operators of (2.2)

$$d_r \propto \oint d\sigma e^{-ir\sigma} \mathbf{d}_K(\sigma).\tag{2.29}$$

Using this formulation of the super-Virasoro constraints it would seem natural to represent them on a Hilbert space whose 'vacuum' state  $|\text{vac}\rangle$  is the *constant 0-form* on loop space, i.e.

$$\partial_{(\mu,\sigma)} |\text{vac}\rangle = 0 = \mathcal{E}_{(\mu,\sigma)} |\text{vac}\rangle \quad \forall \mu, \sigma.\tag{2.30}$$

While this is not the usual  $\text{SL}(2, \mathbb{C})$  invariant vacuum of the closed string, it is precisely the boundary state (2.22)

$$|\text{vac}\rangle = |\mathbf{b}\rangle\tag{2.31}$$

describing the D9 brane.

For the open string NS sector the last relation of (2.22) changes the sign

$$(\psi_r^\mu + i\bar{\psi}_{-r}^\mu) |\mathbf{b}'\rangle = 0, \quad \forall r, \mu \text{ NS sector}\tag{2.32}$$

and now implies that the vacuum is, from the loop space perspective, the formal *volume form* instead of the constant 0-form, i.e. that form annihilated by all differential form multiplication operators:

$$\partial_{(\mu,\sigma)} |\mathbf{b}'\rangle = 0 = \mathcal{E}^{\dagger(\mu,\sigma)} |\mathbf{b}'\rangle \quad \forall \mu, \sigma.\tag{2.33}$$

In finite dimensional flat manifolds of course both are related simply by *Hodge duality*:

$$|b'\rangle = \star |b\rangle . \quad (2.34)$$

So Hodge duality on loop space translates to the NS  $\leftrightarrow$   $R$  transition on the open string sectors.

**Parametrized loop space** It is important to note that we are dealing with the space  $\mathcal{L}(\mathcal{M})$  of *paramerized* loops. More precisely, for our purposes we define this space as the closure of the space of continuous maps  $X$  from the *open* interval  $(0, 2\pi)$  into target space  $\mathcal{M}$ , such that the image is a closed loop with a single point removed:

$$\mathcal{L}(\mathcal{M}) := \overline{\left\{ X : (0, 2\pi) \rightarrow \mathcal{M} \mid \lim_{\epsilon \rightarrow 0} (X(\epsilon) - X(2\pi - \epsilon)) = 0 \right\}} . \quad (2.35)$$

This is a weak form of singling out a base point, i.e. of working on the space of based loops, and it will turn out to be necessary in order to have a sensible notion of reparametrization invariance in the presence of nonabelian Wilson lines along the loops. Precisely the same phenomenon is known from all approaches to non-abelian surface holonomy, and its meaning and implications will be discussed in detail in §3.3 (p.17) after we have derived some formulas in the following sections.

### 2.3.2 Connections on loop space

It is now straightforward to identify the relation between background fields induced by deformations (2.8) and connections on loop space. A glance at (2.27) shows that we have to interpret the term of differential form grade +1 in the polar supersymmetry generator as  $\mathcal{E}^\dagger \hat{\nabla}_\mu^{(\Phi)}$ , where  $\hat{\nabla}_\mu^\Phi$  is a loop space connection (covariant derivative) induced by the target space background field  $\Phi$ .

Indeed, as was shown in [33], one finds for instance that a gravitational background  $G_{\mu\nu}$  leads to  $\hat{\nabla}^{(G)}$  which is just the Levi-Civita connection on loop space with respect to the metric induced from target space. Furthermore, an *abelian* 2-form field background is associated with a deformation operator

$$\mathbf{W}^{(B)} = \oint d\sigma B_{\mu\nu} \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu} \quad (2.36)$$

and leads to a connection

$$\hat{\nabla}_\mu^{(G)(B)} = \hat{\nabla}_\mu^{(G)} - iT B_{\mu\nu} X'^\nu , \quad (2.37)$$

just as expected for a string each of whose points carries  $U(1)$  charge under  $B$  proportional to the length element  $X' d\sigma$ .

## 3. BSCFT deformation for nonabelian 2-form fields

The above mentioned construction can now be used to examine deformations that involve nonabelian 2-forms:

### 3.1 Nonabelian Lie-algebra valued forms on loop space

When gauge connections on loop space take values in nonabelian algebras deformation operators such as  $\exp(\mathbf{W}^{(A)})$  (2.20) and  $\exp(\mathbf{W}^{(B)})$  (2.36) obviously have to be replaced by path ordered exponentiated integrals. The elementary properties of loop space differential forms involving such path ordered integrals are easily derived, and were for instance given in [47, 3].

So consider a differential  $p + 1$  form  $\omega$  on *target space*. It lifts to a  $p + 1$ -form  $\Omega$  on loop space given by

$$\Omega := \frac{1}{(p+1)!} \int_{S^1} \omega_{\mu_1 \dots \mu_{p+1}}(X) \mathcal{E}^{\dagger \mu_1} \dots \mathcal{E}^{\dagger \mu_{p+1}}. \quad (3.1)$$

Let  $\hat{K} = X^{(\mu, \sigma)} \mathcal{E}_{(\mu, \sigma)}$  be the operator of interior multiplication with the reparametrization Killing vector  $K$  (2.28) on loop space. The above  $p + 1$ -form is sent to a  $p$ -form  $\oint(\omega)$  on loop space by contracting with this Killing vector (brackets will always denote the graded commutator):

$$\begin{aligned} \oint(\omega) &:= [\hat{K}, \Omega] \\ &= \frac{1}{p!} \int_{S^1} d\sigma \omega_{\mu_1 \dots \mu_{p+1}} X^{\mu_1} \mathcal{E}^{\dagger \mu_2} \dots \mathcal{E}^{\dagger \mu_{p+1}}. \end{aligned} \quad (3.2)$$

The anticommutator of the loop space exterior derivative  $\mathbf{d}$  with  $\hat{K}$  is just the reparametrization Killing Lie derivative

$$[\mathbf{d}, \hat{K}] = i\mathcal{L}_K \quad (3.3)$$

which commutes with 0-modes of fields of definite reparametrization weight, e.g.

$$[\mathcal{L}, \Omega] = 0. \quad (3.4)$$

It follows that

$$[\mathbf{d}, [\hat{K}, \Omega]] = [\mathcal{L}, \Omega] - [\hat{K}, [\mathbf{d}, \Omega]] \quad (3.5)$$

which implies that

$$[\mathbf{d}, \oint(\omega)] = \oint(-d\omega). \quad (3.6)$$

The generalization to multiple path-ordered integrals

$$\oint(\omega_1, \dots, \omega_n) := \int_{0 < \sigma_{i-1} < \sigma_i < \sigma_{i+1} < \pi} d^n \sigma [\hat{K}, \omega_1](\sigma_1) \dots [\hat{K}, \omega_n](\sigma_n) \quad (3.7)$$

is (see §B.2 (p.31) for the derivation)

$$\begin{aligned} & \left[ \mathbf{d}, \oint(\omega_1, \dots, \omega_n) \right] = \\ & = - \sum_k (-1)^{\sum_{i < k} p_i} \left( \oint(\omega_1, \dots, d\omega_k, \dots, \omega_n) + \oint(\omega_1, \dots, \omega_{k-1} \wedge \omega_k, \dots, \omega_n) \right). \end{aligned} \quad (3.8)$$

This is proposition 1.6 in [47].

Notice that our definition (2.35) of loop space restricts integrations over  $\sigma$  to the complement of the single point  $\sigma = 0 \sim 2\pi$ , so that “boundary” terms  $\omega_1(0) \oint(\omega_2, \dots, \omega_n) \pm \oint(\omega_1, \dots, \omega_{n-1}) \omega_n(2\pi)$  do not appear.

In the light of (2.8) we are furthermore interested in expressions of the form  $U_A(2\pi, 0) \circ \mathbf{d}_K \circ U_A(0, 2\pi)$  where  $U_A$  is the holonomy of  $A$ .

Using

$$\begin{aligned} [\mathbf{d}, U_A(0, 2\pi)] &= \left[ \mathbf{d}, \sum_{n=0}^{\infty} \oint \underbrace{(iA, \dots, iA)}_{n \text{ times}} \right] \\ &= - \sum_{n=0}^{\infty} \sum_k \oint (iA, \dots, iA, iF_A, iA, \dots, iA)_{n \text{ occurrences of } iA, F_A \text{ at } k} \\ &= \int_0^{2\pi} d\sigma U_A(0, \sigma) \left[ iF_A, \hat{K} \right](\sigma) U(\sigma, 2\pi) \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} F_A &= -i(d + iA)^2 \\ &= dA + iA \wedge A \end{aligned} \quad (3.10)$$

is the field strength of  $A$  (which is taken to be hermitean), one finds

$$U_A(2\pi, 0) \circ \mathbf{d} \circ U_A(0, 2\pi) = \mathbf{d} + \int_0^{2\pi} d\sigma U_A(2\pi, \sigma) \left[ iF_A, \hat{K} \right](\sigma) U_A(\sigma, 2\pi). \quad (3.11)$$

The point that will prove to be crucial in the following discussion is that there is an  $A$ -holonomy on *both* sides of the 1-form factor. The operator on the right describes parallel transport with  $A$  from  $2\pi$  to  $\sigma$ , application of  $[\mathbf{d}_A A, \hat{K}]$  at  $\sigma$  and then parallel transport back from  $\sigma$  to  $2\pi$ . Following [3] the abbreviating notation

$$\oint_A(\omega) := \int_0^{2\pi} d\sigma U_A(2\pi, \sigma) \left[ \hat{K}, \omega \right] U_A(\sigma, 2\pi) \quad (3.12)$$

will prove convenient. (But notice that in (3.12) there is also a factor  $U_A(\sigma, 2\pi)$  on the *right*, which does not appear in [3].) Using this notation (3.11) is rewritten as

$$U_A(2\pi, 0) \circ \mathbf{d} \circ U_A(0, 2\pi) = \mathbf{d} - i \oint_A(F_A). \quad (3.13)$$

This expression will prove to play a key role in the further development. In order to see why this is the case we now turn to the computation and discussion of the connection on loop space which is induced by the nonabelian 2-form background.

### 3.2 Nonabelian 2-form field deformation

With the above considerations it is now immediate how to incorporate a nonabelian 2-form in the target space of a boundary superconformal field theory on the worldsheet. The direct generalization of (2.20) and (2.36) is obviously the deformation operator

$$\exp(\mathbf{W})^{(A)(B)\text{nonab}} = \text{P exp} \left( \int_0^{2\pi} d\sigma \left( iA_\mu X'^\mu + \frac{1}{2} \left( \frac{1}{T} F_A + B \right)_{\mu\nu} \mathcal{E}^{\dagger\mu} \mathcal{E}^{\dagger\nu} \right) \right) \quad (3.14)$$

for non-abelian and hermitean  $A$  and  $B$ . (P denotes path ordering) Note that this is indeed reparametrization invariant on  $\mathcal{L}(\mathcal{M})$  (2.35) and that there is no trace in (3.14), so that this must act on an appropriate bundle<sup>2</sup>, which is naturally associated with a stack of  $N$  branes (*cf.* pp. 3-4 of [48]).

According to §2.3.2 (p.13) the loop space connection induced by this deformation operator is given by the term of degree +1 in the deformation of the superconformal generator (2.27). Using (3.11) this is found to be

$$\begin{aligned} \exp(-\mathbf{W})^{(A)(B)\text{nonab}} \circ \mathbf{d}_K \circ \exp(\mathbf{W})^{(A)(B)\text{nonab}} &= \mathbf{d} + iT \oint_A (B) \\ &\quad + (\text{terms of grade } \neq 1), \end{aligned} \quad (3.15)$$

where the notation (3.12) is used.

The second term  $iT \oint_A (B)$  is the nonabelian 1-form connection on loop space which is induced by the target space 2-form  $B$ . Note that the terms involving the  $A$ -field strength  $d_A A$  coming from the  $X'$  term and those coming from the  $\mathcal{E}^\dagger \mathcal{E}^\dagger$  term in (3.15) mutually cancel.

The connection (3.15) is essentially that given in [3], instead that here the  $U_A$  holonomy acts from both sides, as in the expressions given in [27, 36]. It will be shown below that this is crucial for the correct gauge invariance on target space. In [31] it will be discussed in detail how this very form of the gauge connection makes the non-abelian 2-form formalism derived here from SCFT deformations equivalent to that of 2-group theory [29, 28].

There is a simple way to understand the form of (3.15) heuristically: A point particle couples to a 1-form, a string to a 2-form. Imagine the worldsheet foliated into spacelike hyperslices. Each *point* on these is similar to a point particle coupled to that 1-form which is obtained by contracting the 2-form with the tangent vector to the slice at that point. In the abelian case all these contributions can simply be summed up and so one obtains a corresponding 1-form on loop space, namely (2.37). But really, as noted in [3] in general one has to be more careful, since elements of fibers at different points can not be compared.

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<sup>2</sup>In this paper however only local aspects of such a bundle play a role.

Instead, we can relate the 1-forms at each point of the string by parallel transporting them with respect to some 1-form connection  $A$  to a given (arbitrary) origin. This is precisely what is accomplished by the  $U_A$  factors in (3.15).

It should be emphasized that even though the construction (3.15) involves a unitary transformation of the superconformal generators this does *not* imply that the 2-form gauge connection (3.15) is trivial up to gauge. It is crucial that the deformation of the term  $iT\mathcal{E}_\mu X'^\mu$  in  $\mathbf{d}_K$  (2.27) also contributes to the gauge connection. This way, the result is *not* a unitary transform of the loop space exterior derivative  $\mathbf{d}$ , but only of its  $K$ -deformed relative  $\mathbf{d}_K$ . Of course precisely this effect could already be observed in simpler examples of abelian backgrounds discussed in [33].

Still, we will see below that precisely the case where there is *no* contribution from  $iT\mathcal{E}_\mu X'^\mu$  will turn out to be the most interesting one.

Before discussing gauge transformations and equations of motion it is in order to have a careful look at a certain technicality:

### 3.3 Parallel transport to the base point

One point deserves further attention: Note that in (3.14) we did *not* include a trace over the path-ordered exponential. The reason for that is quite simple: If we included a trace, making the formerly group-(representation)-valued expression a scalar, this could not give rise to an exterior derivative on loop space which is covariant with respect to the given non-abelian gauge group, simply because its connection term would be locally a loop space 1-form taking scalar values instead of non-abelian Lie-algebra values. This way the relation to non-abelian surface holonomy would be completely lost and one could not expect the associated SCFT to describe any non-abelianness of the background.

There is a further indication that not taking the trace is the correct thing to do: Recall from the discussion at the beginning of this section (*cf.* table 1) that we should expect states of strings in a non-abelian 2-form background to have a single Chan-Paton-like degree of freedom, i.e. to carry one fundamental index of the gauge group. This implies that in particular boundary states carry such an index and hence any deformation operator acting on them must accordingly act on that index. This is precisely what an un-traced generalized Wilson line as in (3.14) does.

However, by not taking the trace in (3.14) the point  $\sigma = 0 \sim 2\pi$  on the string/loop becomes a preferred point in a sense. The object (3.14) is reparameterization invariant only under those reparameterizations that leave the point  $\sigma = 0 \sim 2\pi$  where it is. Commuting it with a general generator of  $\sigma$ -reparameterizations produces two “boundary” terms (even though we do not really have any boundary at  $\sigma = 0 \sim 2\pi$  on the closed string) at  $\sigma = 0$  and  $\sigma = 2\pi$  which only mutually cancel when traced over.

This might at first sight appear as a problem for our proposed way to study worldsheet theories in non-abelian 2-form backgrounds. After all, reparameterization invariance on the string must be preserved by any reasonable physical theory. But a closer look at the general theory of non-abelian surface holonomy indicates how the situation should clarify:

As soon as a non-abelian 2-form is considered and any notion of non-abelian surface-holonomy associated with it, a fundamental question that arises is at which *point* the non-abelian holonomy of a given surface “lives” [3]. It must be associated with some point, because it really lives in a given fiber of a gauge bundle over target space, and it has to be specified in which one. This issue becomes quite rigorously clarified in the 2-group description of surface holonomy [28, 49, 29], whose basic mechanism is summarized from a physical point of view in [31]. In the 2-group description non-abelian surface holonomy is a functor that assigns 2-group elements to “bigons”, which are surfaces with two special points on their boundary (a “source” and a “target” vertex). It is a theorem (e.g. proposition 5 in [29]) that every 2-group uniquely comes from a “crossed module” (a tuple of two groups, one associated with our 1-form  $A$  the other with the 2-form  $B$  together with a way for  $A$  to act on  $B$  by derivations) in such a way that when composing any two bigons, their total surface holonomy is obtained by parallel transporting (with respect to the 1-form  $A$ ) group elements from the individual “source” vertices to the common source vertex of the combined bigon. This theorem shows that it is inevitable to associate surface holonomy with “preferred” points.

On the other hand, these points are not absolutely preferred. One can choose any other point on the boundary of the given bigon as a the source vertex. This is done by, again, parallel transporting the surface holonomy from the original source vertex to the new one. A more detailed description of this and related facts of 2-group theory is beyond the scope of this paper, but can be found in [31], which relates 2-group theory as described in [28, 49, 29] to the loop space formalism used here and in [27]. Indeed, it is shown there that the preferred point  $\sigma = 0 \sim 2\pi$  of (3.14) is to be identified with the source vertex in 2-group theory.

That this is true can indeed be seen quite clearly from the expression (3.15) given below, which we find for the non-abelian covariant loop space exterior derivative (*cf.* with (3.12) for the notation used there) obtained by deforming the ordinary loop space exterior derivative with (3.14) (as described in detail below). Indeed, the loop space covariant exterior derivative to be discussed in the following involves integrating the non-abelian 2-form  $B$  over the string, while parallel transporting its value back from every point to  $\sigma = 2\pi$ . As mentioned above, and as already noted in [3], this parallel transport back to a common reference point is necessary in order to obtain a well defined element of a fiber in a non-abelian gauge bundle.

It should be plausible from these comments, and is proven in detail in [31], that, given non-abelian differential forms  $A$  and  $B$ , the surface holonomy computed using the covariant loop space connections discussed here coincide with those computed by 2-group theory, when the former are integrated over loops in *based* loop space, i.e. the space of all loops with a common point in target space.

But this should finally clarify the appearance of the preferred point  $\sigma = 0$  in (3.14) also from the point of view of worldsheet reparameterization invariance: Whenever we work with a non-abelian 2-form, the worldsheet surpercharges, which are generalized exterior derivatives on loop space [33], must be regarded as operators on *based* loop space (for some given base point  $x$ ), in terms of which, for the reasons just discussed, all computations

must necessarily take place. However, the choice of this base point is arbitrary, as surface holonomies computed with differing base points are related simply by parallel transport between the two base points.

The restriction to based loop space may seem like a drastic step from the string worldsheet point of view, even though the above general considerations make it seem to be quite inevitable. On the other hand, the boundary state formalism that we will be concerned with in the following involves spherical closed string worldsheets at tree level, and these are precisely the closed curves in based loop space (*cf.* [27]).

Apart from loop space and 2-group methods a further conceptual framework for surface holonomy is the theory of *gerbes* [50]. Gerbes have been applied with much avail to string theory in the presence of 2-form fields, but so far mostly in the abelian case. Approaches to construct a theory of non-abelian gerbes can be found in [16, 8, 7], but, to the best of our knowledge, it is not yet understood how to compute non-abelian surface holonomy using gerbes. One proposal is [36] which makes use of precisely the parallel transport to common base points known from 2-group theory and found from the SCFT deformations used here (but without taking consistency conditions on the uniqueness of surface holonomy into account). On the other hand, the notion of abelian surface holonomy from abelian gerbes is well understood [50, 51]. It can be expressed [52] in precisely the loop space language used here where it, too, works on based loop space.

Even with so many compelling formal reasons to have a preferred point on the string in the presence of non-abelian 2-form background it would be nice to also have a more physically motivated interpretation of this phenomenon. One aspect that might be relevant is the observation in [53] that at least in certain 6-dimensional theories certain strings can be regarded as coming from M-theory membranes that stretch between points identified under a  $Z_r$  orbifold action. A similar mechanism was discussed in [54] in the context of compactifications of type IIA strings down to six dimensions. Maybe the preferred worldsheet point  $\sigma = 0$  that we found above to be necessary for the coupling to the non-abelian 2-form field has to be identified as attached to one of the points identified under the orbifold action.

Given all this, the conceptual issues that we encounter in our worldsheet perspective approach to non-abelian 2-form field backgrounds are somewhat unfamiliar, but are quite in accordance with both the general features expected of the target space theory as well as general facts known about non-abelian 2-gauge theory. While it of course by no means completely solve the issue of superstring propagation in non-abelian 2-form backgrounds, the present approach is hoped to illuminate some crucial aspects of any complete theory that does so, and in particular of its worldsheet formulation.

With that technical point discussed we now turn to the issue of gauge transformations of the loop space connection (3.15) and how this relates to gauge transformations on target space.

### 3.4 2-form gauge transformations

In a gauge theory with a nonabelian 2-form one expects the usual gauge invariance

$$\begin{aligned} A &\mapsto U A U^\dagger + U(dU^\dagger) \\ B &\mapsto U B U^\dagger \end{aligned} \tag{3.16}$$

together with some nonabelian analogue of the infinitesimal shift

$$\begin{aligned} A &\mapsto A + \Lambda \\ B &\mapsto B - d_A \Lambda + \dots \end{aligned} \tag{3.17}$$

familiar from the abelian theory.

With the above results, it should be possible to derive some properties of the gauge invariances of a nonabelian 2-form theory from loop space reasoning. That's because on loop space (3.15) is an ordinary *1-form* connection. The ordinary 1-form gauge transformations of that loop space connection should give rise to something like (3.16) and (3.17) automatically.

Indeed, *global* gauge transformations of (3.15) on loop space give rise to (3.16), while infinitesimal gauge transformations on loop space give rise to (3.17), but with correction terms that only have an interpretation on loop space.

More precisely, let  $U(X) = U$  be any *constant* group valued function on (a local patch of) loop space and let  $V(X) : \mathcal{M} \rightarrow G$  be such that  $\lim_{\epsilon \rightarrow 0} V(X)(X(\epsilon)) = V(X)(X(2\pi - \epsilon)) = U$ , then (see §B (p.30) for the details)

$$U \left( \oint_A (B) \right) U^\dagger + U(dU^\dagger) = \oint_{A'} (B') \tag{3.18}$$

with

$$\begin{aligned} A' &= V A V^\dagger + V(dV^\dagger) \\ B' &= V B V^\dagger, \end{aligned} \tag{3.19}$$

which reproduces (3.16).

If, on the other hand,  $U$  is taken to be a *nonconstant* infinitesimal gauge transformation with a 1-form gauge parameter  $\Lambda$  of the form

$$U(X) = 1 - i \oint_A (\Lambda) \tag{3.20}$$

then

$$U \left( \oint_A (B) \right) U^\dagger + U(dU^\dagger) = \oint_{A+\Lambda} (B + d_A B) + \dots \tag{3.21}$$

The first term reproduces (3.17), but there are further terms which do not have analogs on target space.

The reason for this problem can be understood from the boundary deformation operator point of view:

Let

$$\text{P exp} \left( i \int_0^{2\pi} R \right) = \lim_{N=1/i\epsilon \rightarrow \infty} (1 + i\epsilon R(0)) \cdots (1 + i\epsilon R(\epsilon 2\pi)) \cdots (1 + i\epsilon R(2\pi)) \quad (3.22)$$

be the path ordered integral over some object  $R$ . Then a small shift  $R \rightarrow R + \delta R$  amounts to the “gauge transformation”

$$\text{P exp} \left( i \int_0^{2\pi} R \right) \rightarrow \text{P exp} \left( i \int_0^{2\pi} R \right) \left( 1 + i \oint_R \delta R \right) \quad (3.23)$$

to first order in  $\delta R$ . Notice how, using the definition of  $\oint_R$  given in (3.12), the term  $\oint_R \delta R$  inserts  $\delta R$  successively at all  $\sigma$  in the preceding Wilson line.

So it would seem that  $U = 1 - i \oint_R \delta R$  is the correct unitary operator, to that order, for the associated transformation. But the problem is that  $R$  is in general not purely bosonic, but contains fermionic contributions. These spoil the ordinary interpretation of the above  $U$  as a gauge transformation.

This means that an ordinary notion of gauge transformation is obtained if and only if the fermionic contributions in (3.14) disappear, which is the case when

$$B = -\frac{1}{T} F_A. \quad (3.24)$$

Then

$$\exp(\mathbf{W})^{(A)(B=-\frac{1}{T}F_A)} = \text{P exp} \left( \int_0^{2\pi} d\sigma i A_\mu X'^\mu(\sigma) \right) \quad (3.25)$$

is the pure  $A$ -Wilson line and the corresponding gauge covariant exterior derivative on loop space is

$$\mathbf{d}^{(A)(B)} = \mathbf{d} - i \oint_A (F_A), \quad (3.26)$$

as in (3.13). Now the transformation

$$\begin{aligned} A &\mapsto A + \Lambda \\ B &\mapsto B - \frac{1}{T} (d\Lambda + iA \wedge \Lambda + i\Lambda \wedge A) \end{aligned} \quad (3.27)$$

is correctly, to first order, induced by the loop space gauge transformation

$$U(X) = 1 - i \oint_A (\Lambda). \quad (3.28)$$

One can check explicitly that indeed

$$\begin{aligned}
U \circ (\mathbf{d} - i \oint_A (F_A)) \circ U^\dagger &= \mathbf{d} - i \oint_A (F_A + d\Lambda + iA \wedge \Lambda + i\Lambda \wedge A) + \mathcal{O}(\Lambda^2) \\
&\quad - \int_{\sigma_1 \geq \sigma_2} U_A(2\pi, \sigma_1) (-iF_A)(\sigma_1) U_A(\sigma_1, \sigma_2) i\Lambda_\mu X'^\mu(\sigma_2) U_A(\sigma_2, 2\pi) \\
&\quad + \int_{\sigma_2 \geq \sigma_1} U_A(2\pi, \sigma_1) i\Lambda_\mu X'^\mu(\sigma_1) U_A(\sigma_1, \sigma_2) (-iF_A)(\sigma_2) U_A(\sigma_2, 2\pi) \\
&\quad + \oint_A (-iF_A) \oint_A (i\Lambda) + \oint_A (-i\Lambda) \oint_A (-iF_A) \\
&\quad + \mathcal{O}(\Lambda^2) \\
&= \mathbf{d} - i \oint_{A+\Lambda} (F_{A+\Lambda}) + \mathcal{O}(\Lambda^2) , \tag{3.29}
\end{aligned}$$

as it must be.

One can furthermore check that the loop space curvature of the connection (3.26) *vanishes*, as follows also directly from (3.13):

$$\left( \mathbf{d} - i \oint_A (F_A) \right)^2 = 0 . \tag{3.30}$$

A flat connection on loop space implies that the surface holonomy associated with tori in target space is trivial. As discussed at the end of §2.2 (p.10), this makes sense, since the nonabelian connection should not couple to the closed string without boundary, due to lack of Chan-Paton factors or anything that could play their role.

The above shows that for flat loop space connections the expected gauge invariances (3.16) and (3.17) of 2-form gauge theory do hold without problematic correction terms and that apparently a consistent physical picture is obtained.

The same result has been obtained recently in [28] using the theory of 2-groups as introduced in [29].

### 3.5 Flat connections on loop space and surface holonomy

In the light of the flatness (3.30) of the connection (3.26) in this section some general aspects of flat connections on loop space and their relation to parameterization invariant surface holonomies are discussed in the following.

Denote by  $\mathcal{LL}(\mathcal{M})$  the space of parameterized loops in loop space. The holonomy of the loop space connection  $\oint_A(B)$  around these loops in loop space is a map

$$H : \mathcal{LL}(\mathcal{M}) \rightarrow G , \tag{3.31}$$

where  $G$  is the gauge group. This computes the *surface holonomy* of the (possibly degenerate) unbounded surface in target space associated with a given loop-loop in  $\mathcal{LL}(\mathcal{M})$ .

In general, there are many points in  $\mathcal{LL}(\mathcal{M})$  that map in a bijective way to the same surface in  $\mathcal{M}$  and that are related by reparameterization. Only if the function  $H$  takes the

same value on all these points does the loop space connection  $\oint_A(B)$  induce a well-defined surface holonomy in target space.

In the case considered above, where, locally at least,  $\mathcal{M} = \mathbb{R}^D$ , all loops in  $\mathcal{L}(\mathcal{M})$  are contractible. This means that when  $\oint_A(B)$  is flat,  $H$  maps *all of*  $\mathcal{L}\mathcal{L}(\mathcal{M})$  to the identity element in  $G$ . In this case the surface holonomy is therefore trivially well defined, since all closed surfaces represented by points on  $\mathcal{L}\mathcal{L}(\mathcal{M})$  are assigned the same surface holonomy,  $H = 1$ .

This is the case that has been found here to arise from boundary state deformations. It implies that unbounded closed string worldsheets do not see the nonabelian background. But worldsheets having a boundary that come from cutting open those surfaces corresponding to points in  $\mathcal{L}\mathcal{L}(\mathcal{M})$  do. Is their surface holonomy also well defined?

For the special connection (3.26) it is. This follows from the fact that this is just the trivial connection, which assigns the unit to everything, gauge transformed with (3.26). Under a gauge transformation on loop space the surface holonomies of bounded surfaces coming from open curves in  $\mathcal{L}(\mathcal{M})$  are simply multiplied from the left by the gauge transformation function (3.26) evaluated at one boundary  $B_1$  and from the right by its inverse evaluated at the other boundary  $B_2$ :

$$H_{\text{bounded}} = \exp(\mathbf{W})(B_1) \exp(\mathbf{W})^{-1}(B_2) . \quad (3.32)$$

But  $\exp(\mathbf{W})$  is reparameterization invariant on the loop, as long as the preferred point  $\sigma = 0$  is unaffected, at which the Wilson loop is open (untraced). Therefore the surface holonomy induced by (3.26) on open curves in  $\mathcal{L}(\mathcal{M})$  depends (only) on the position of the preferred point  $\sigma = 0$ .

In the boundary state formalism, the point  $\sigma = 0$  has to be identified with the insertion of the open string state which propagates on half the closed string worldsheet. So this dependence appears to make sense.

In the case when  $\pi_2(\mathcal{M})$  is nontrivial it is not as obvious to decide if a flat connection on loop space associates unique surface holonomy with surfaces in  $\mathcal{M}$ .

A sufficient condition for this to be true is that any two points in  $\mathcal{L}\mathcal{L}(\mathcal{M})$  which map to the same given surface can be connected by continuously deforming the corresponding loop on  $\mathcal{L}(\mathcal{M})$ . This is not true for nondegenerate toroidal surfaces in  $\mathcal{M}$ . But it is the case for spherical surfaces, for which the loop on  $\mathcal{L}(\mathcal{M})$  must begin and end at an infinitesimal loop. All slicings of the sphere are continuously deformable into each other, corresponding to a deformation of a loop on  $\mathcal{L}(\mathcal{M})$ , under which the holonomy of a flat connection is invariant.

Therefore even with nontrivial  $\pi_2(\mathcal{M})$  a flat connection on loop space induces a well defined surface holonomy on topologically spherical surfaces in  $\mathcal{M}$ .

Whether the same statement remains true for toroidal surfaces is not obvious. But actually for open string amplitudes at tree level in the boundary state formalism spherical surfaces are all that is needed.

Due to these considerations it is an interesting task to try to characterize all flat connections on parameterized loop space. The connection (3.26) is obtained by loop space gauge transformations from the trivial connection. Are there other flat connections?

Some flat connections on loop space were investigated in the context of integrable systems in [27]. The authors of that paper used the same general form (3.15) of the connection on loop space that dropped out from deformation theory in our approach. They then demanded that the  $A$ -curvature vanishes,  $F_A \stackrel{!}{=} 0$ , and checked that for this special case an  $A$ -covariantly constant  $B$ , as well as an  $A$ -closed  $B$  which furthermore takes values in an abelian ideal, is sufficient to flatten the connection  $\oint_A(B)$ . In the notation used here, this can be seen as follows:

The curvature of the general connection (3.15) is

$$\begin{aligned} \mathbf{F}^{(A)(B)} &:= -i \left( \mathbf{d}^{(A)(B)} \right)^2 \\ &= -i \left( \mathbf{d} + iT \oint_A(B) \right)^2 \\ &= T \oint_A(d_A B) + iT^2 \oint_A(B) \oint_A(B) + (\text{terms proportional to } F_A). \end{aligned} \quad (3.33)$$

For vanishing  $A$ -field strength this reduces to

$$\mathbf{F}^{(A)(B)} \stackrel{F_A=0}{=} T \oint_A(d_A B) + iT^2 \oint_A(B) \oint_A(B). \quad (3.34)$$

It is easy to see under which conditions both terms on the right hand side vanish by themselves (while it seems hardly conceivable that there are conditions under which these two terms cancel mutually without each vanishing by themselves), namely the first term vanishes when  $d_A B = 0$ , while the second term vanishes when the components of the 1-form  $\oint_A(B)$  mutually commute.

This is for instance the case when  $B$  takes values in an abelian ideal or if  $B$  is  $A$ -covariantly constant and all components of  $B$  at a given point commute. These are the two conditions discussed in §3.2 of [27].

Since these two cases correspond to abelian  $B$  we have to think of them as special cases of the theory of abelian 2-form fields. To obtain a scenario with nonabelian  $B$  from these one can again, as we did for the trivial connection, make a gauge transformation (3.26) on loop space with respect to the holonomy of yet another 1-form connection, not necessarily a flat one. This way one obtains two further classes of flat connections on loop space.

Finally it should be noted that we haven't ruled out the possibility that there are non-flat connections on loop space which induce a well defined surface holonomy. (Of course for the abelian case all connections of the form (2.37) induce well defined surface holonomy.) Our main point in §3.2 (p.16) was merely that boundary state deformation theory leads to supersymmetry generators which incorporate a flat connection.

Before discussing this further in the concluding section it pays to first see what the worldsheet theory has to say about the equations of motion of the nonabelian background field. This is the content of the next section.

### 3.6 Background equations of motion

It was demonstrated in [34, 35] that the conditions under which the operator (2.21) for *abelian*  $A$ , *without* any normal ordering, is well defined when acting on the boundary state  $|b\rangle$  (2.31) describing a bare D9 brane, is equivalent to the background equations of motion of  $A$ , at least up to second order.

A similar statement should be true for the nonabelian generalization (3.14) that we are concerned with here. Due to the consistency condition (3.24) we need only consider the case where the worldsheet fermions  $\mathcal{E}^\dagger$  are all canceled by the presence of the  $B$ -field, so that the background equations of motion for the nonabelian  $B$ -field should arise as the condition for the canceling of divergences in the application of (3.25) to  $|b\rangle$ .

This task is greatly simplified by working with gauge connections  $A$  which are *constant* on spacetime. Due to standard arguments (e.g. [55]) this should be no restriction of generality but it greatly simplifies the computation of divergences, since the number of contractions of worldsheet fields is very much reduced.

Namely in the case of constant  $A$ , i.e.  $\partial_\mu A_\nu = 0$ , the only divergences come from terms of the form  $X'^\mu(\kappa)X'^\nu(\sigma)|b\rangle$ . Using the mode expansion (2.23) as well as the boundary condition (2.22) this contraction is seen to produce

$$X'^\mu(\kappa)X'^\nu(\sigma)|b\rangle = \alpha' \eta^{\mu\nu} \sum_{n>0} n \cos(n(\sigma - \kappa)) |b\rangle + :X'^\mu(\kappa)X'^\nu(\sigma): |b\rangle, \quad (3.35)$$

where  $\alpha' = 1/2\pi T$ .

When this expression is inserted into the expansion of (3.25):

$$\begin{aligned} \text{P exp} \left( i \int_0^{2\pi} d\sigma A_\mu X'^\mu(\sigma) \right) |b\rangle &= |b\rangle + i A_\mu \underbrace{\int_0^{2\pi} d\sigma X'^\mu(\sigma)}_{=0} |b\rangle \\ &\quad - A_\mu A_\nu \int_{0 < \sigma_1 < \sigma_2 < 2\pi} d\sigma_1 d\sigma_2 X'^\mu(\sigma_1) X'^\nu(\sigma_2) |b\rangle \\ &\quad - i A_\mu A_\nu A_\lambda \int_{0 < \sigma_1 < \sigma_2 < \sigma_3 < 2\pi} d\sigma_1 d\sigma_2 d\sigma_3 X'^\mu(\sigma_1) X'^\nu(\sigma_2) X'^\lambda(\sigma_3) |b\rangle \\ &\quad + \dots \end{aligned} \quad (3.36)$$

one immediately sees that the  $A^2$  term does not produce any divergence so that the first nontrivial case is the  $A^3$  term. A simple calculation yields

$$\int_{0 < \sigma_1 < \sigma_2 < \kappa} d\sigma_1 d\sigma_2 \cos(n(\sigma_1 - \sigma_2)) = \int_{\kappa < \sigma_1 < \sigma_2 < 2\pi} d\sigma_1 d\sigma_2 \cos(n(\sigma_1 - \sigma_2)) = -\frac{1}{2} \int_{0 < \sigma_1 < \kappa < \sigma_2 < 2\pi} d\sigma_1 d\sigma_2 \cos(n(\sigma_1 - \sigma_2)), \quad (3.37)$$

so that the divergence of the  $A^3$  term is proportional, at each point  $\kappa$ , to

$$\alpha' A_\mu A_\nu A_\lambda \left( \eta^{\mu\nu} X'^\lambda(\kappa) - 2\eta^{\mu\lambda} X'^\nu(\kappa) + \eta^{\nu\lambda} X'^\mu(\kappa) \right) = \alpha' [A^\mu, [A_\mu, A_\lambda]] X'^\lambda(\kappa). \quad (3.38)$$

This vanishes precisely if

$$[A^\mu, [A_\mu, A_\lambda]] = 0. \quad (3.39)$$

Assuming that the restriction to constant gauge connections does not affect the generality of this result we hence find that at first order the condition for the well-definedness of the deformation operator (3.36) is equivalent to

$$\operatorname{div}_A F_A = 0, \quad (3.40)$$

which are of course just the equations of motion of Yang-Mills theory. The full equations of motion for the nonabelian 2-form background found this way are hence

$$\begin{aligned} B &= -\frac{1}{T} F_A \\ \operatorname{div}_A F_A = 0 &= \operatorname{div}_A B. \end{aligned} \quad (3.41)$$

This might appear not to be surprising. However, from the point of view of several approaches to the topic to 2-form gauge theories found in the literature (e.g. [28, 20]) it may look odd, because these equations of motion are invariant under the first order gauge transformation (3.16) but not under that at second order (3.17).

But the discussion in §3.4 (p.20) should clarify this: Both first and second order gauge transformations are symmetries of the *flat* connections on the space of (closed!) loops and hence of the closed string. (For non-flat connections on loop space we found that no consistent formulation is possible at all.) But due to the flatness condition the coupling of the closed string to the nonabelian background fields is trivial, as it should be.

The open string does couple nontrivially to the nonabelian 2-form background, but since the boundary of the disc diagram attached to the brane couples to  $A$ , there is no symmetry under  $A \rightarrow A + \Lambda$ , and from the heuristic picture of string physics there should not.

In this sense it seems like the worldsheet theory deformed by a nonabelian Wilson line.

## 4. Summary and Conclusion

We have demonstrated how a nonabelian connection on loop space

$$\mathbf{d} + iT \oint_A (B) = \mathbf{d} + it \int_0^{2\pi} U_A(2\pi, \sigma) [\hat{K}, B](\sigma) U_A(\sigma, 2\pi) \quad (4.1)$$

can be read off from certain deformations of the worldsheet SCFT by a generalized Wilson line along the string, and how formal consistency conditions on gauge transformations of this loop space connection lead to the relation

$$\frac{1}{T} F_A + B = 0, \quad (4.2)$$

between the 1-form field strength  $F_A$  and the 2-form  $B$ .

This consistency condition was already recently derived in [28] using the theory of 2-groups. There it appeared mainly as an obstacle for writing down “interesting” Lagrangians for higher gauge theories.

Still, by “cutting open” the closed string, which in our formalism means that the original deformation of the closed string SCFT translates into a deformation of a boundary state, a nontrivial coupling of the open string to the nonabelian 2-form background is obtained.

We have calculated the equations of motion of this nonabelian open string background by canceling divergences in the deformed boundary state. The result was, at lowest non-trivial order, the ordinary equations of motion of Yang-Mills theory with respect to the 1-form connection  $A$ , together with the similar equation for  $B$ , implied by the constraint (4.2).

As discussed in [28], this constraint prevents these equations of motion to follow from a simple generalization of the Yang-Mills action to 2-forms. Therefore precisely how the above fits into the framework of “higher gauge theory” remains to be seen.

Conversely, the considerations presented here should show that some aspects of such 2-form gauge theories can be understood by studying the target space theory which is *implied* by certain worldsheet theories.

What is certainly missing, however, is a good understanding of the nature of the backgrounds described by such worldsheet theories.

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## A. Boundary state formalism

As a background for §2.2 (p.10) this section summarizes basic aspects of boundary conformal field theory (as discussed for instance in [56, 57, 58]).

### A.1 BCFTs

Given a conformal field theory on the complex plane (with coordinates  $z, \bar{z}$ ) we get an associated ('descendant') *boundary conformal field theory* (BCFT) on the upper half plane (UHP),  $\text{Im}(z) > 0$ , by demanding suitable boundary condition on the real line. The only class of cases well understood so far is that where the chiral fields  $W(z), \bar{W}(\bar{z})$  can be analytically continued to the real line  $\text{Im}(z) = 0$  and a local automorphism of the chiral algebra exists, the *gluing map*  $\Omega$ , such that on the boundary the left- and right-moving fields are related by

$$W(z) = \Omega \bar{W}(\bar{z}), \quad \text{at } z = \bar{z}. \quad (\text{A.1})$$

In particular  $\Omega$  always acts trivially on the energy momentum current

$$\Omega \bar{T}(\bar{z}) = \bar{T}(\bar{z}) \quad (\text{A.2})$$

so that

$$T(z) = \bar{T}(\bar{z}), \quad \text{at } z = \bar{z}, \quad (\text{A.3})$$

which ensures that no energy-momentum flows off the boundary.

This condition allows to introduce for every chiral  $W, \bar{W}$  the single chiral field

$$W(z) = \begin{cases} W(z) & \text{for } \text{Im}(z) \geq 0 \\ \Omega(\bar{W})(\bar{z}) & \text{for } \text{Im}(z) < 0 \end{cases} \quad (\text{A.4})$$

defined in the entire plane. (This is known as the 'doubling trick'.)

### A.2 Boundary states

Since it is relatively awkward to work with explicit constraints it is desirable to find a framework where the boundary condition on fields at the real line can be replaced by an operator insertion in a bulk theory without boundary.

Imagine an open string propagating with both ends attached to some D-brane. The worldsheet is topologically the disk (with appropriate operator insertions at the boundary). This disk can equivalently be regarded as the half sphere glued to the brane. But from this point of view it represents the worldsheet of a closed string with a certain source at the brane. Therefore the open string disk correlator on the brane is physically the same as a closed string emission from the brane with a certain source term corresponding to the open string boundary condition. The source term at the boundary of the half sphere can be represented by an operator insertion in the full sphere. The state corresponding to this vertex insertion is the *boundary state*.

In formal terms this heuristic picture translates to the following procedure:

First map the open string worldsheet to the sphere, in the above sense. By stereographic projection, the sphere is mapped to the plane and the upper half sphere which represents the open string worldsheet disk gets mapped to the complement of the unit disk in the plane. Denote the complex coordinates on this complement by  $\zeta, \bar{\zeta}$  and let the open string worldsheet time  $\tau = -\infty$  be mapped to  $\zeta = 1$  and  $\tau = +\infty$  mapped to  $\zeta = -1$  (so that the open string propagates 'from right to left' in these worldsheet coordinates). With  $z, \bar{z}$  the coordinates on the UHP this corresponds to  $z = 0 \mapsto \zeta = 1$  and  $z = \infty \mapsto \zeta = -1$ . The rest of the boundary of the string must get mapped to the unit circle, which is where the string is glued to the brane. An invertible holomorphic map from the UHP to the complement of the unit disk with these features<sup>3</sup> is

$$\zeta(z) := \frac{1 - iz}{1 + iz}. \quad (\text{A.6})$$

For a given boundary condition  $\alpha$  the boundary state  $|\alpha\rangle$  is now defined as the state corresponding to the operator which, when inserted in the sphere, makes the correlator of some open string field  $\Phi$  on the sphere equal to that on the UHP with boundary condition  $\alpha$ :

$$\langle \Phi^{(\text{H})}(z, \bar{z}) \rangle_\alpha = \left( \frac{\partial \zeta}{\partial z} \right)^h \left( \frac{\partial \bar{\zeta}}{\partial \bar{z}} \right)^{\bar{h}} \langle 0 | \Phi^{(\text{P})}(\zeta, \bar{\zeta}) | \alpha \rangle. \quad (\text{A.7})$$

Noting that on the boundary we have

$$\frac{\partial \zeta}{\partial z} = -i\zeta, \quad \text{at } z = \bar{z} \Leftrightarrow \zeta = 1/\bar{\zeta} \quad (\text{A.8})$$

the gluing condition (A.1) becomes in the new coordinates

$$\begin{aligned} \left( \frac{\partial \zeta}{\partial z} \right)^h W(\zeta) &= \left( \frac{\partial \bar{\zeta}}{\partial \bar{z}} \right)^{\bar{h}} \Omega \bar{W}(\bar{\zeta}) \\ \Leftrightarrow W(\zeta) &= (-1)^h \zeta^{2h} \Omega \bar{W}(\bar{\zeta}), \quad \text{at } \zeta = 1/\bar{\zeta}. \end{aligned} \quad (\text{A.9})$$

In the theory living on the plane this condition translates into a constraint on the boundary state  $|\alpha\rangle$ :

$$\begin{aligned} 0 &\stackrel{!}{=} \langle 0 | \cdots \sum_{n=-\infty}^{\infty} \left( W_n \zeta^{-n-h} - (-1)^h \zeta^{-2h} \Omega \bar{W}_n \zeta^{n+h} \right) | \alpha \rangle \\ &= \langle 0 | \cdots \sum_{n=-\infty}^{\infty} \left( W_n \zeta^{-n-h} - (-1)^h \Omega \bar{W}_n \zeta^{n-h} \right) | \alpha \rangle, \quad \forall \zeta = 1/\bar{\zeta}, \end{aligned} \quad (\text{A.10})$$

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$$|\zeta|^2 = \frac{1 + |z|^2 + 2\text{Im}(z)}{1 + |z|^2 - 2\text{Im}(z)} \geq 1 \quad \text{for } \text{Im}(z) \geq 0 \quad (\text{A.5})$$

i.e.

$$\left(W_n - (-1)^h \Omega \bar{W}_{-n}\right) |\alpha\rangle = 0, \quad \forall n \in \mathbb{N}. \quad (\text{A.11})$$

Since  $\Omega \bar{T} = \bar{T}$  holds for all BCFTs this implies in particular that one always has

$$(L_n - \bar{L}_{-n}) |\alpha\rangle = 0 \quad \forall n, \quad (\text{A.12})$$

which says that  $|\alpha\rangle$  is invariant with respect to reparametrizations of the spatial worldsheet variable  $\sigma$  parameterizing the boundary (*cf.* for instance section 3 of [33]).

## B. Computations

This section compiles some calculations which have been omitted from the main text.

### B.1 Global gauge transformations on loop space

For completeness the following gives a derivation of (3.18), which is essentially nothing but the ordinary proof of gauge invariance of the Wilson line:

Consider any path-ordered Integral  $I_R$  over some arbitrary object  $R$  from, say 0 to 1:

$$I_R := \lim_{N=1/\epsilon \rightarrow \infty} [(1 + \epsilon R(0)) (1 + \epsilon R(\epsilon)) (1 + \epsilon R(2\epsilon)) \cdots (1 + \epsilon R(1))]. \quad (\text{B.1})$$

Let  $U = U(\sigma)$  be a unitary function on the loop which does not depend on the embedding field  $X$ . By acting with it on the endpoints of the above path ordered integral we get

$$\begin{aligned} U(0) I_R U^\dagger(1) &:= \lim_{N=1/\epsilon \rightarrow \infty} U(0) \left[ (1 + \epsilon R(0)) U^\dagger(\epsilon) U(\epsilon) (1 + \epsilon R(\epsilon)) U^\dagger(2\epsilon) U(2\epsilon) \cdots (1 + \epsilon R(1)) \right] U^\dagger(1) \\ &= \lim_{N=1/\epsilon \rightarrow \infty} \left[ \left( U(0) U^\dagger(\epsilon) + \epsilon U(0) R(0) U^\dagger(\epsilon) \right) \cdots \right. \\ &\quad \left. \cdots \left( U((N-1)\epsilon) U^\dagger(1) + \epsilon U((N-1)\epsilon) R(1) U^\dagger(1) \right) \right] \\ &= \lim_{N=1/\epsilon \rightarrow \infty} \left[ \left( 1 + \epsilon U U^\dagger + \epsilon U R U^\dagger \right)(0) \cdots \left( 1 + \epsilon U U^\dagger + \epsilon U R U^\dagger \right)(1) \right]. \end{aligned} \quad (\text{B.2})$$

If the term in  $R$  proportional to  $X'^\mu$  is identified with  $A_\mu$  this gives the gauge transformation  $A \mapsto U(dU^\dagger) + UAU^\dagger$  for  $A$  and  $B \mapsto UBU^\dagger$  for the remaining components of  $R$ .

The same applies in the other  $\sigma$ -direction:

$$\begin{aligned} U(1) J_R U^\dagger(0) &:= \lim_{N=1/\epsilon \rightarrow \infty} U(1) \left[ (1 - \epsilon R(1)) U^\dagger((N-1)\epsilon) U((N-1)\epsilon) \right. \\ &\quad \left. \cdots (1 + \epsilon R(\epsilon)) U^\dagger((N-2)\epsilon) U((N-2)\epsilon) (1 - \epsilon R(0)) \right] U^\dagger(0) \\ &= \lim_{N=1/\epsilon \rightarrow \infty} \left[ \left( U(1) U^\dagger((N-1)\epsilon) - \epsilon U(1) R(1) U^\dagger((N-1)\epsilon) \right) \cdots \right. \\ &\quad \left. \cdots \left( U(\epsilon) U^\dagger(0) - \epsilon U(\epsilon) R(0) U^\dagger(0) \right) \right] \\ &= \lim_{N=1/\epsilon \rightarrow \infty} \left[ \left( 1 - \epsilon U U^\dagger - \epsilon U R U^\dagger \right)(1) \cdots \left( 1 - \epsilon U U^\dagger - \epsilon U R U^\dagger \right)(0) \right]. \end{aligned} \quad (\text{B.3})$$

In particular, it follows that

$$U(0) \left( \mathbf{d} + \oint_A (B) \right) U^\dagger(0) = \mathbf{d} + \oint_{UAU^\dagger + U(U^\dagger)'} (UBU^\dagger). \quad (\text{B.4})$$

This directly gives equation (3.18) used in the main text.

## B.2 The exterior derivative of path-ordered loop space forms

The action of the loop space exterior derivative in (3.8) is derived as follows:

$$\begin{aligned} & \left[ \mathbf{d}, \oint(\omega_1, \dots, \omega_n) \right] \\ &= \sum_k (-1)^{1 + \sum_{i < k} p_i} \oint(\omega_1, \dots, d\omega_k, \dots, \omega_n) \\ & \quad + \sum_k (-1)^{\sum_{i < k} p_i} \int_{0 < \sigma_{i-1} < \sigma_i < \sigma_{i+1} < \pi} d^n \sigma \left[ \hat{K}, \omega_1 \right](\sigma_1) \cdots (\omega_k)' \cdots \left[ \hat{K}, \omega_n \right](\sigma_n) \\ &= \sum_k (-1)^{1 + \sum_{i < k} p_i} \oint(\omega_1, \dots, d\omega_k, \dots, \omega_n) \\ & \quad + \sum_k (-1)^{1 + \sum_{i < k} p_i} \int_{0 < \sigma_{i-1} < \sigma_i < \sigma_{i+1} < \pi} d^n \sigma \left[ \hat{K}, \omega_1 \right](\sigma_1) \cdots \left( \left[ \hat{K}, \omega_{k-1} \right] \omega_k - (-1)^{p_{k-1}} \omega_{k-1} \left[ \hat{K}, \omega_k \right] \right) (\sigma_k) \cdots \left[ \hat{K}, \omega_n \right](\sigma_{n-1}) \\ &= \sum_k (-1)^{1 + \sum_{i < k} p_i} \left( \oint(\omega_1, \dots, d\omega_k, \dots, \omega_n) + \oint(\omega_1, \dots, \omega_{k-1} \wedge \omega_k, \dots, \omega_n) \right). \end{aligned}$$

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